Arrangement & Generalized Voronoi Diagrams

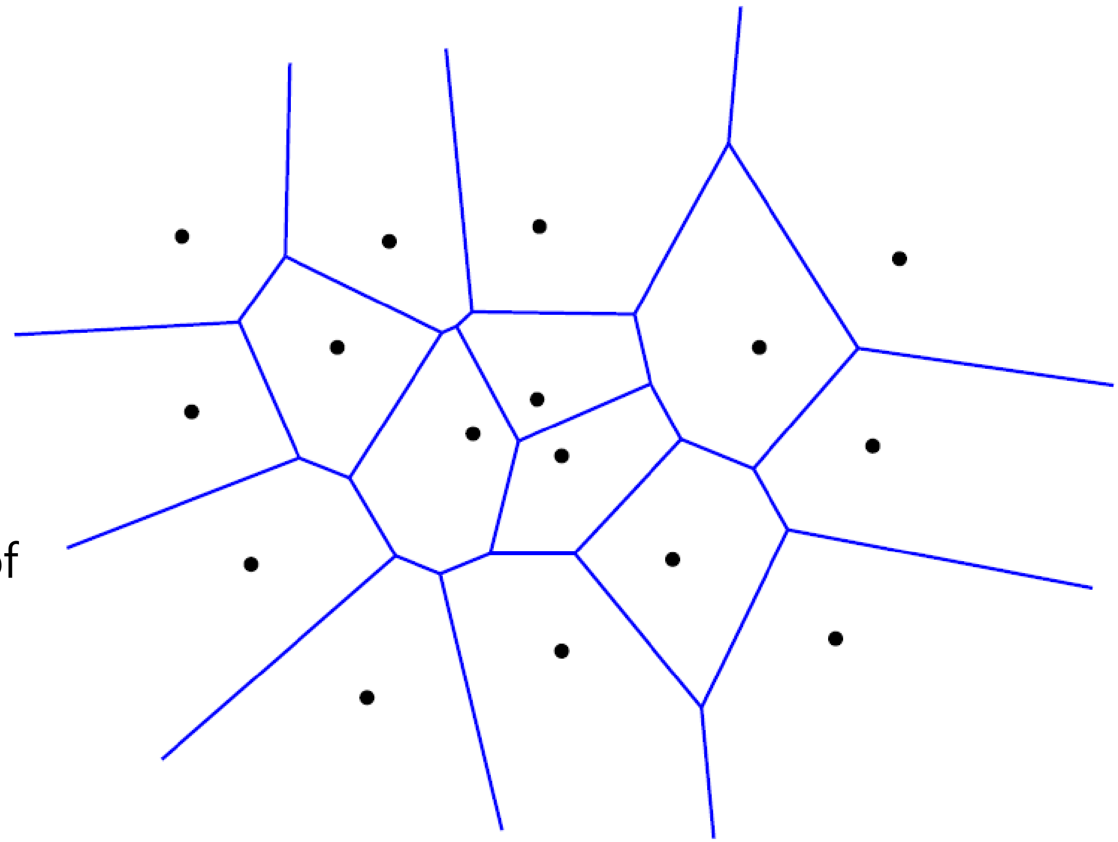
Geometric Computing

Voronoi diagram

Voronoi diagram of a set of points(called ***sites***) :

Union of the Voronoi cells for each site

Voronoi cell of a site – set of points closest to that site



Generalization

- Generalized sites
  - line segments, circles, curves, polygon, area
- Various distances & spaces
  - Manhattan, L_p-norm, weighted distances, power distance, convex distance, nice metrics, time distance, …
  - In higher dimensions, on a sphere, on a cone, in a polygon, with obstacles, …
- Higher-order VDs
  - Order-k VDs & Farthest VDs

Cones

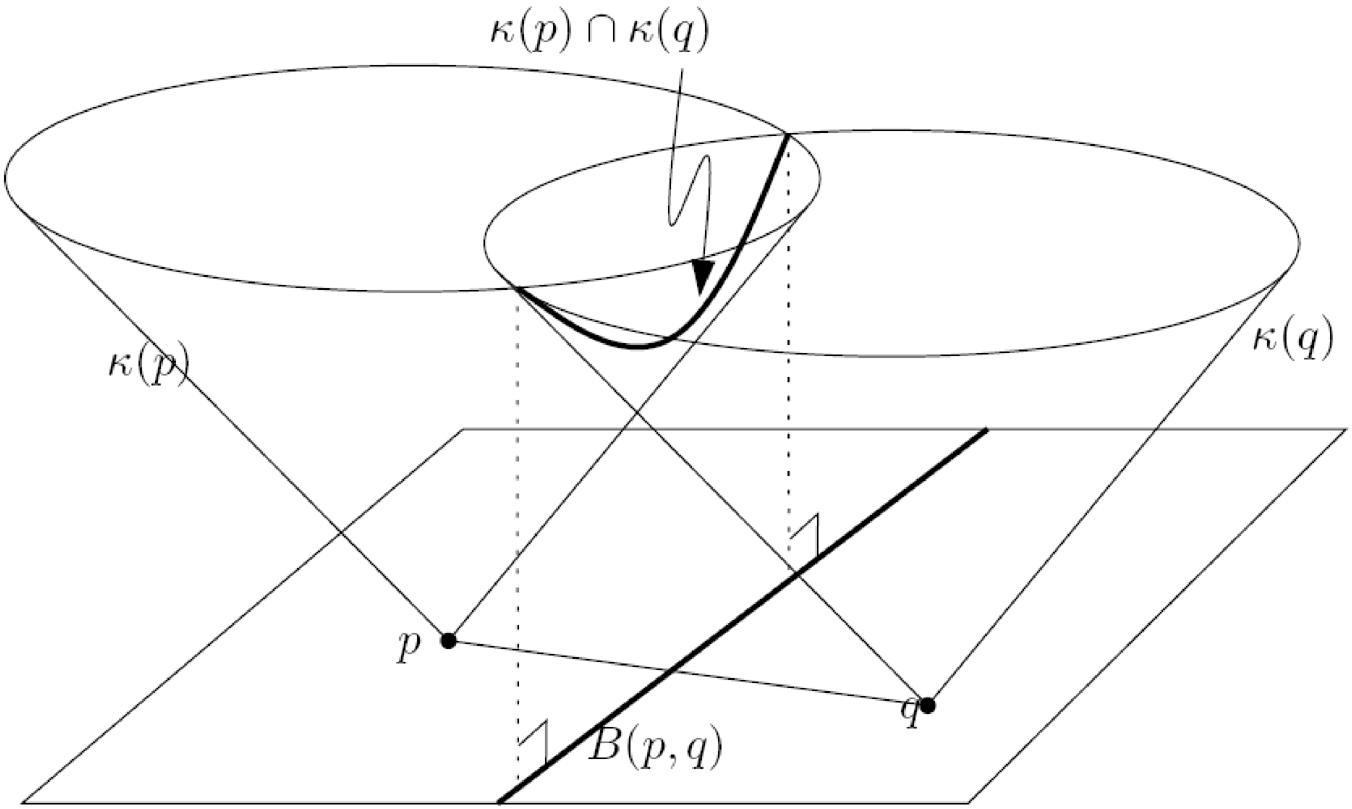
- We are given
  - D : a distance function in $(\mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R})$
  - S : points in $\mathbf{R}^d$

- Lift distances to higher space

$$\kappa(p) : x_{d+1} = D(x, p)$$

Dirichlet 1850 & Voronoi 1908

Cones for Points on $\mathbb{E}^2$



Weighted VDs

- Consider Weighted Sites : (p, w_p)

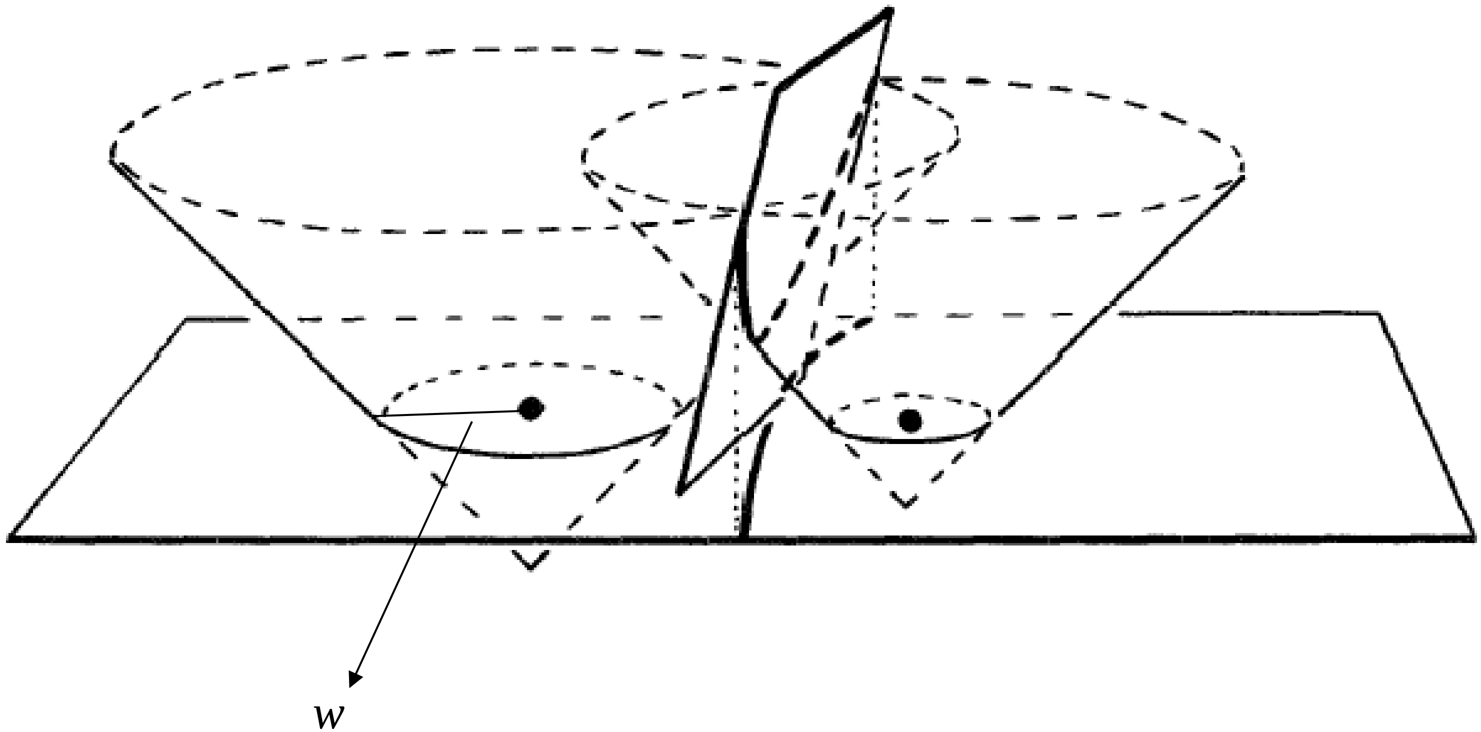
  – Additively Weighted Distance

  $$d_a(q,p) = d(q,p) - w_p$$

  – Multiplicatively Weighted Distance

  $$d_m(q,p) = d(q,p)/w_p$$

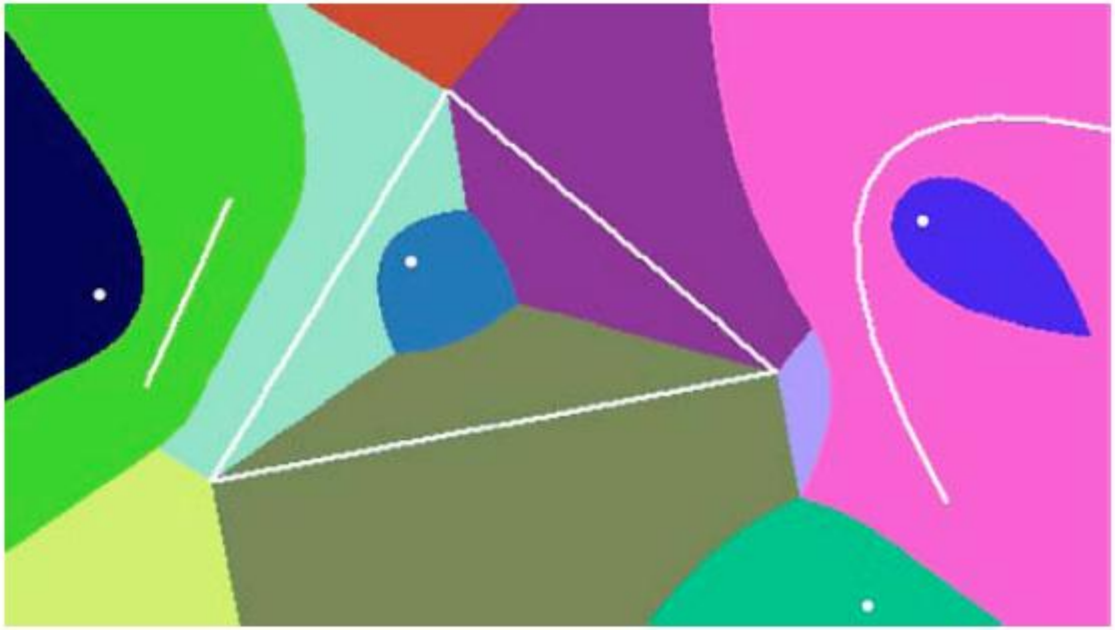
Cones for Additively-Weighted Points



Applying this idea

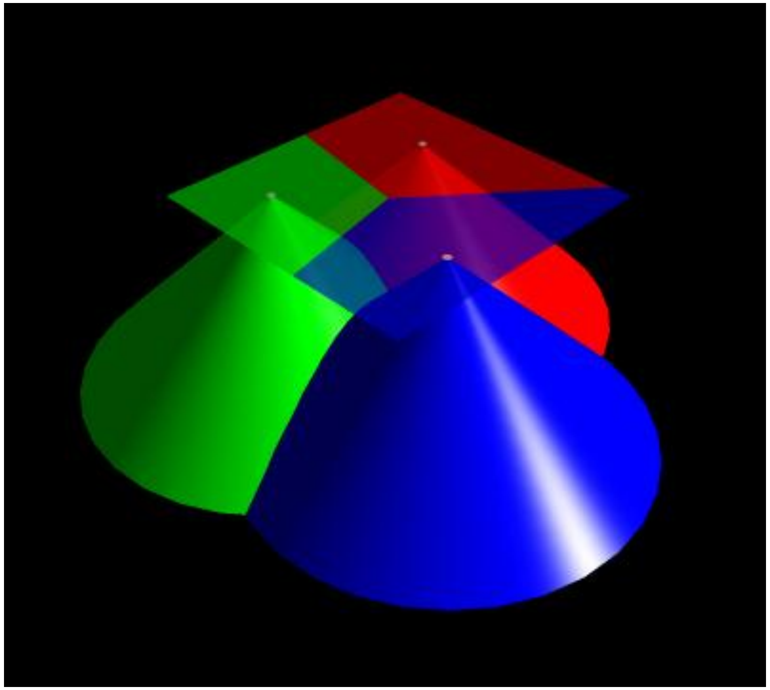
- *Fast Computation of Generalized Voronoi Diagrams Using Graphics Hardware*, Hoff, Culver, Keyser, Lin, and Manocha, Siggraph 1999.

Discrete approximation of the generalized VD of 4 points, a line, a triangle and a curve

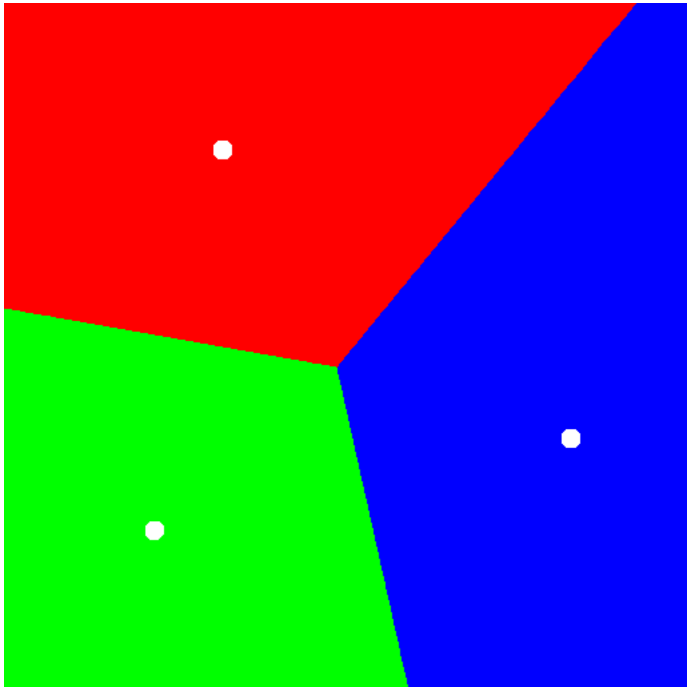


Cone Drawing

To visualize Voronoi diagram for points in 2D…



Perspective, 3/4 view

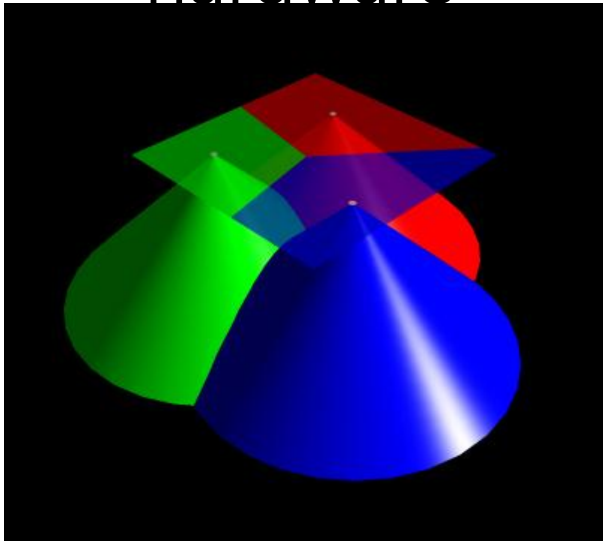


Parallel, top view

Dirichlet 1850 & Voronoi 1908

Graphics Hardware Acceleration

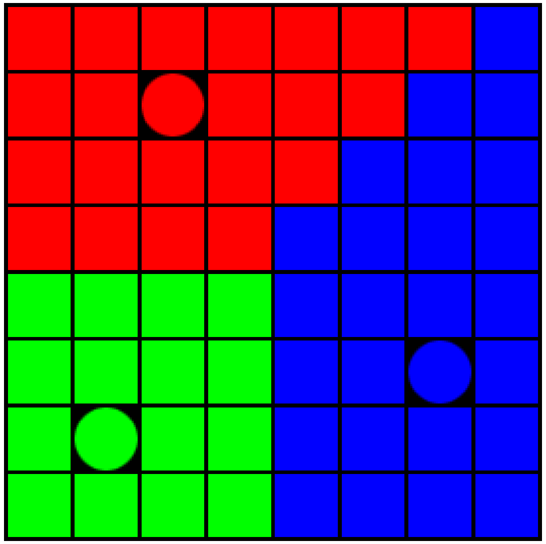
Simply rasterize the cones using graphics hardware



Haeberli90, Woo97

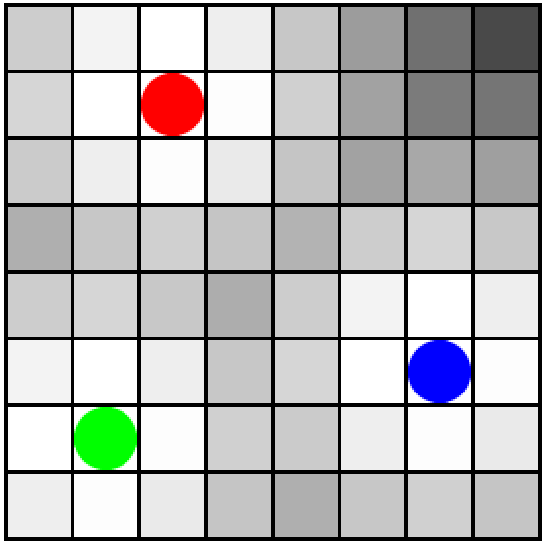
2-part discrete Voronoi diagram representation

Color Buffer



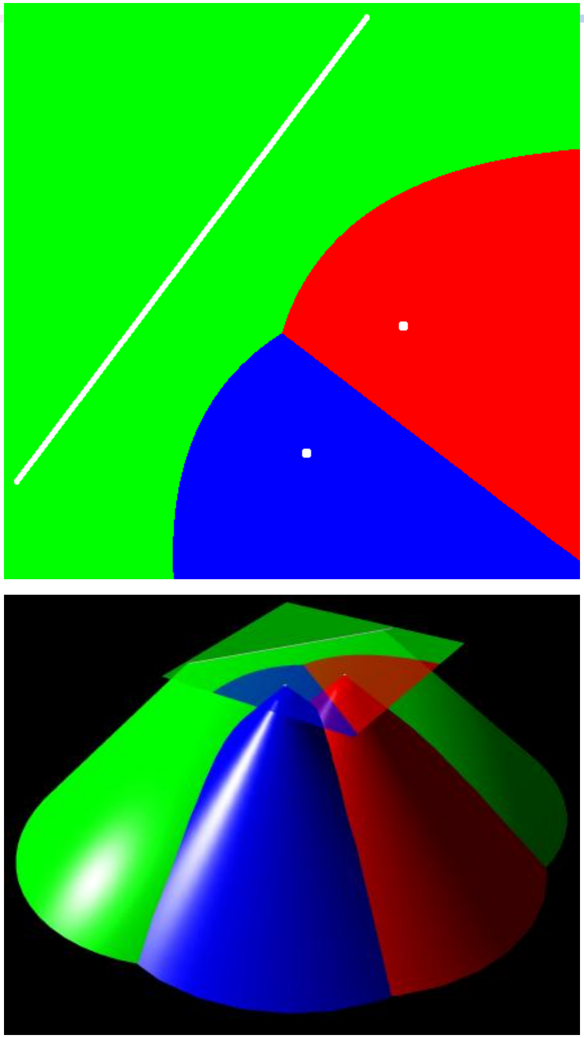
Site IDs

Depth Buffer

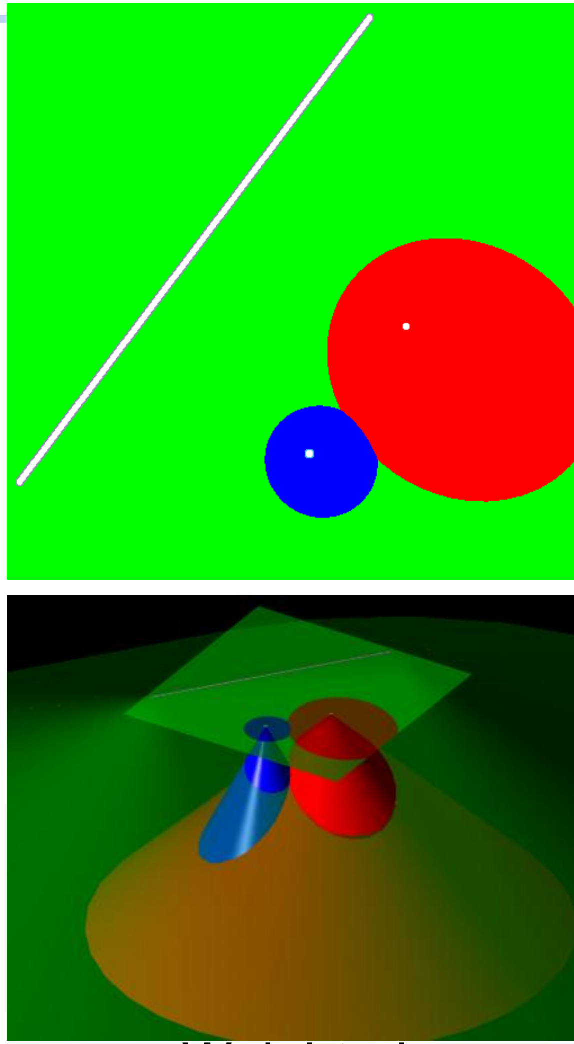


Distance

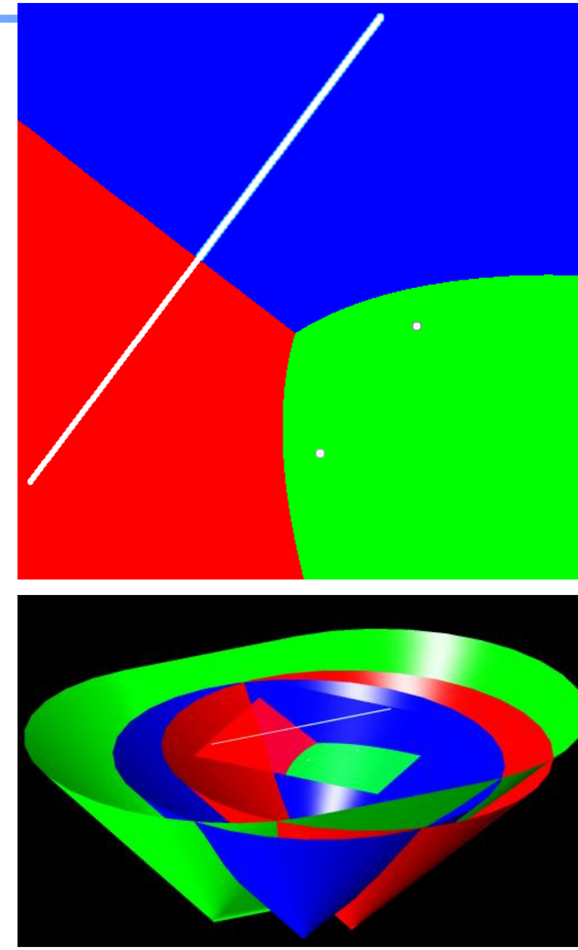
Weighted and Farthest Distance



Nearest



Weighted



Farthest

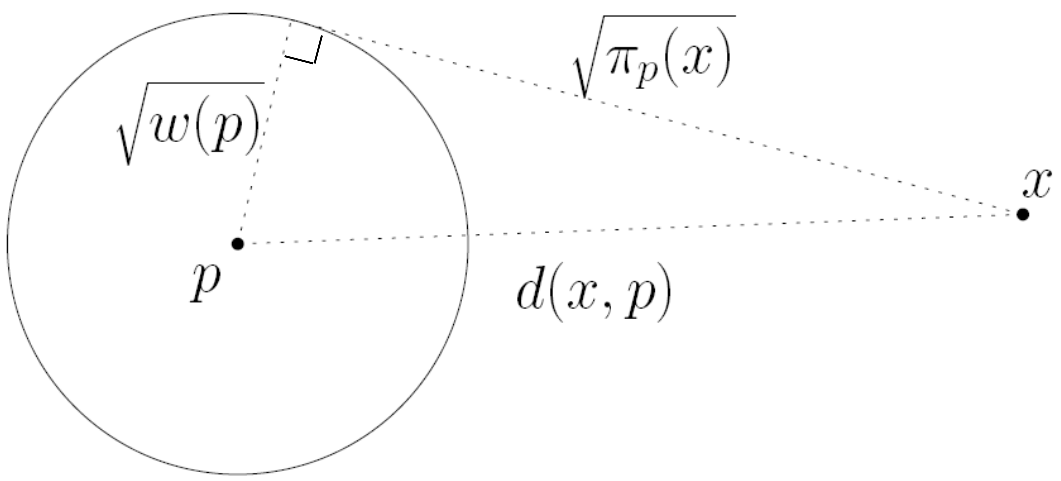
Power Diagram

- By F. Aurenhammer in 1987
- Given a set of weighted points
- A power diagram can be computed from a convex hull in one dimension higher
- AWVD and MWVD can be obtained from a power diagram in one dimension higher

Power Diagram

- Power distance to p

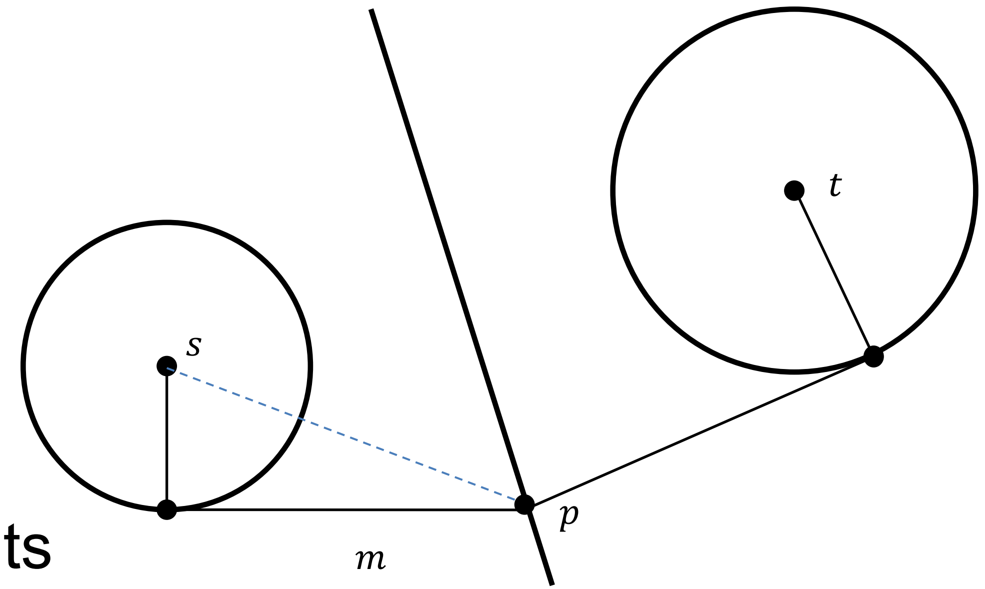
$$\pi_p(x) = d(x,p)^2 - w(p)$$



Power Diagram

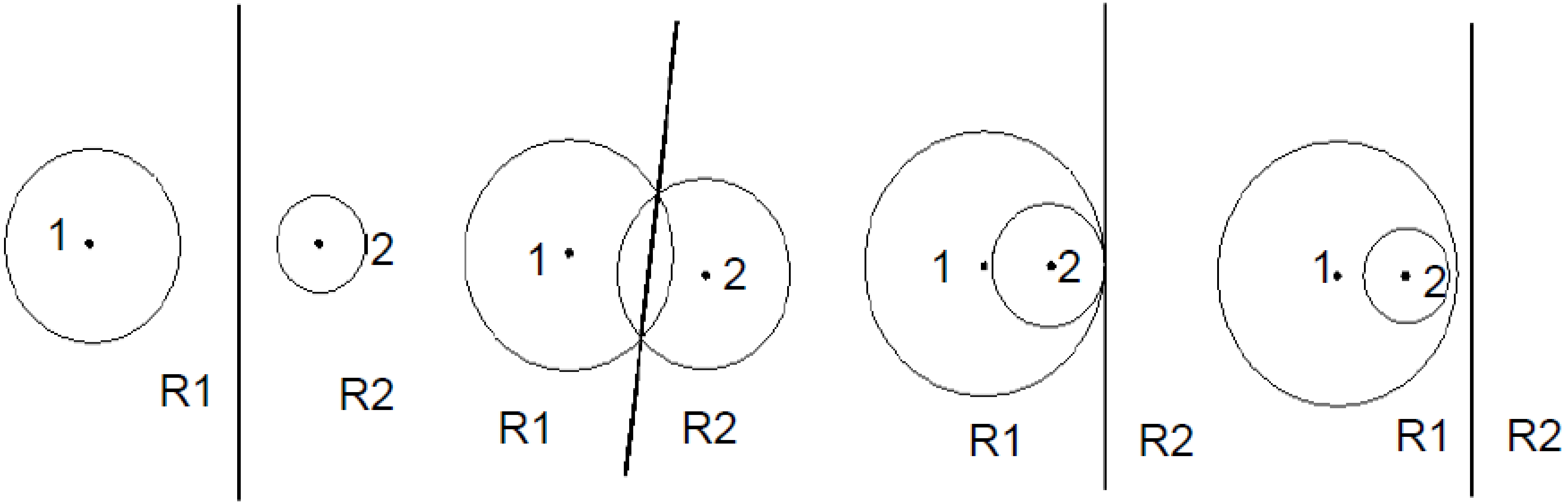
- $|S| = 2$

$$\begin{aligned} & d_s(p) = d_t(p) \\ \Leftrightarrow\ & d(s,p)^2 - w_s^2 \\ & \quad = d(t,p)^2 - w_t^2 \\ \Leftrightarrow\ & d(s,p)^2 - d(t,p)^2 \\ & \quad = w_s^2 - w_t^2 \\ & \quad = constant \end{aligned}$$

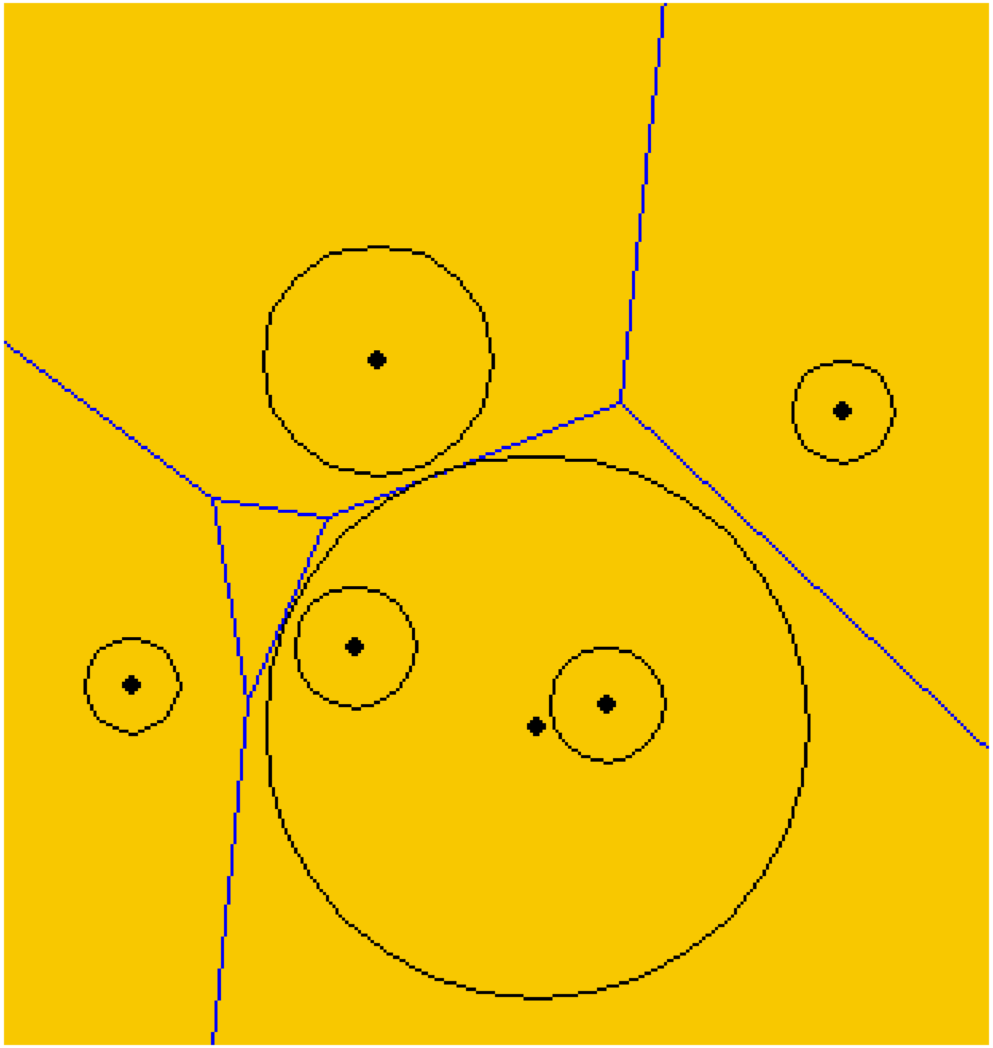


in 2D, the locus of the points equidistant from two weighted points is a straight line.

Power Diagram

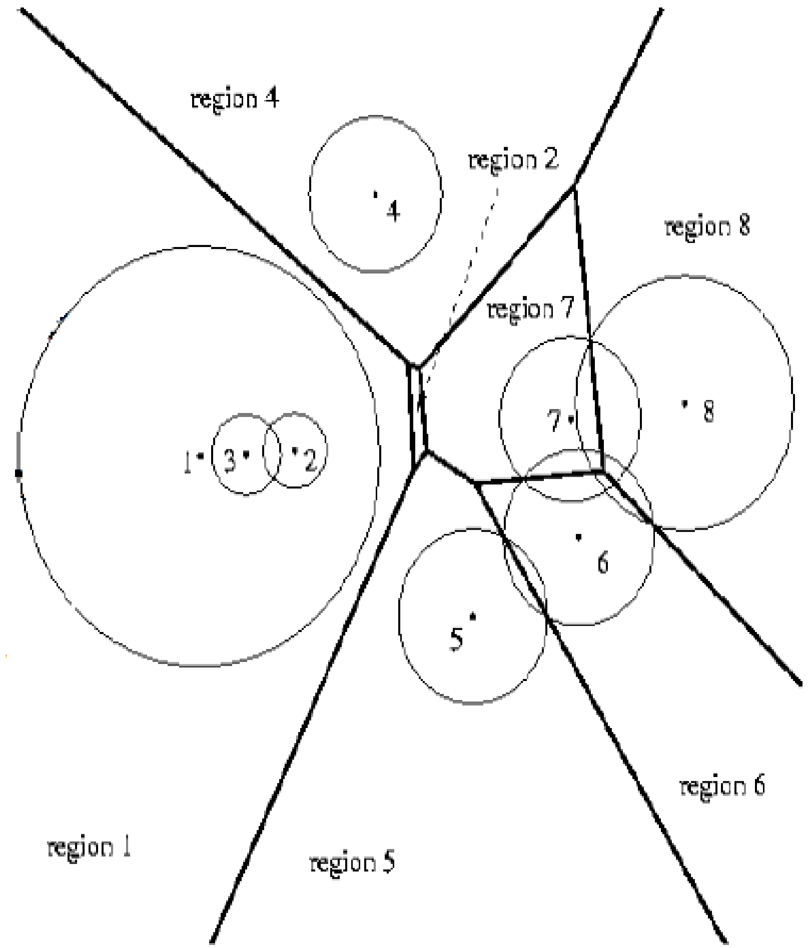


Power Diagram



Power Diagram

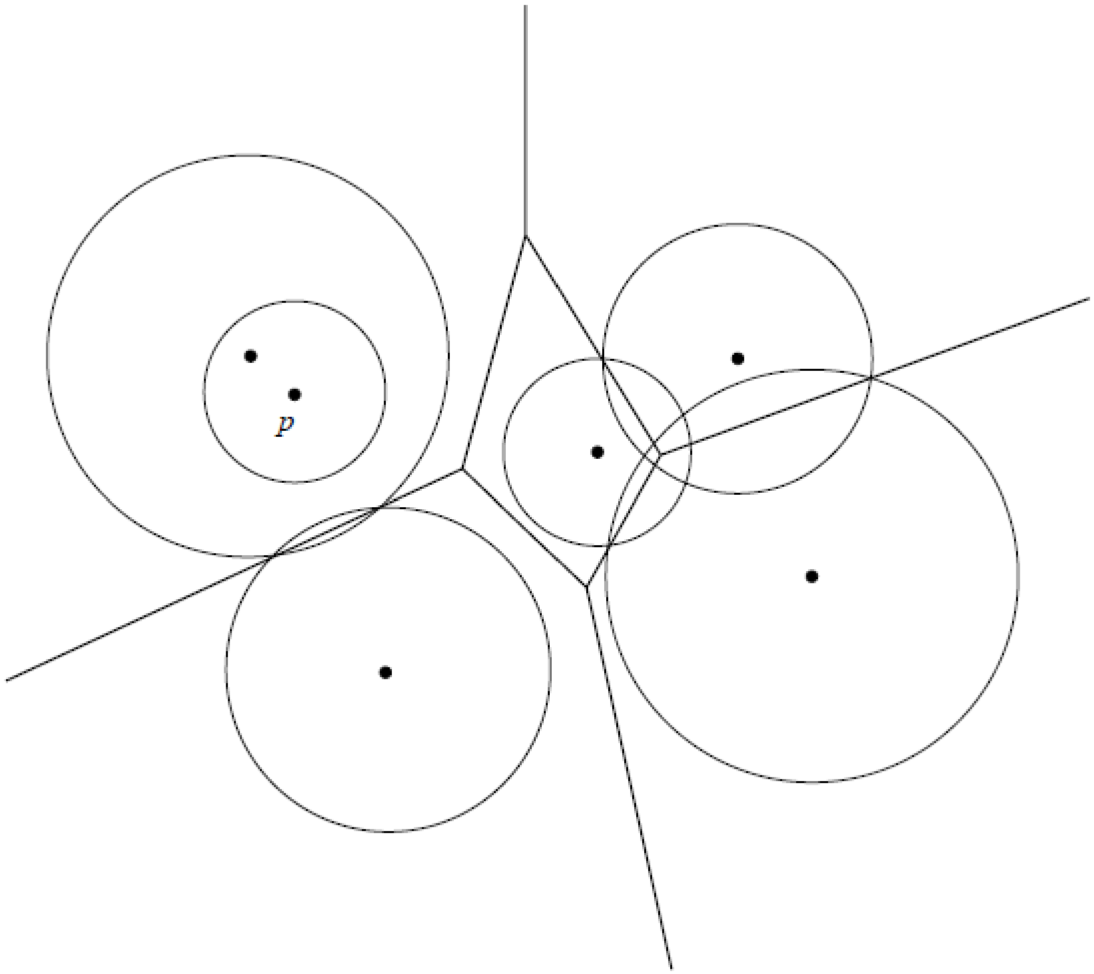
- A power region may be empty
- A power region of p may not contain the point p
- A point on the convex hull boundary has an unbounded or an empty region



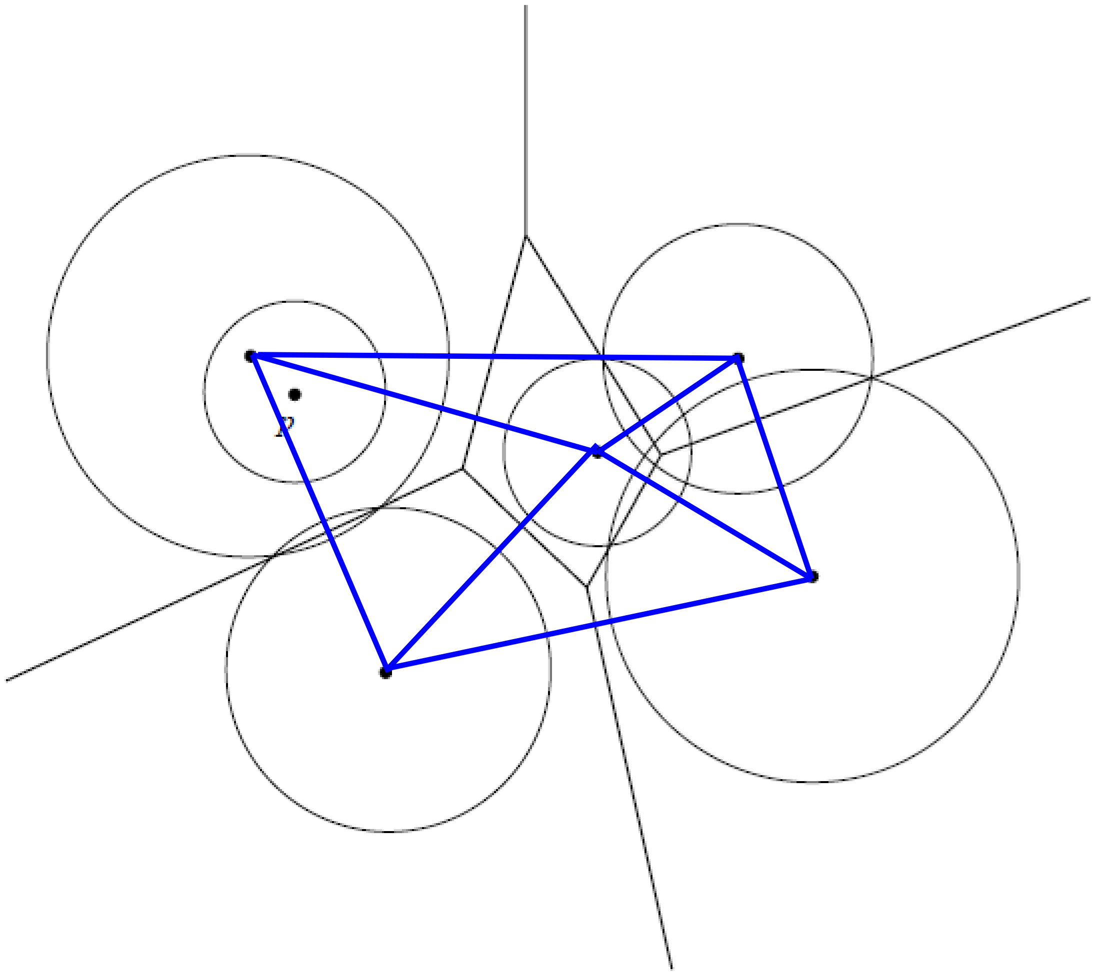
Power Diagram

- dual graph of the Power Diagram
    - *"regular triangulation"*

Power Diagram



Power Diagram



Lifting-up the Power Diagram

- Cones for Power distances

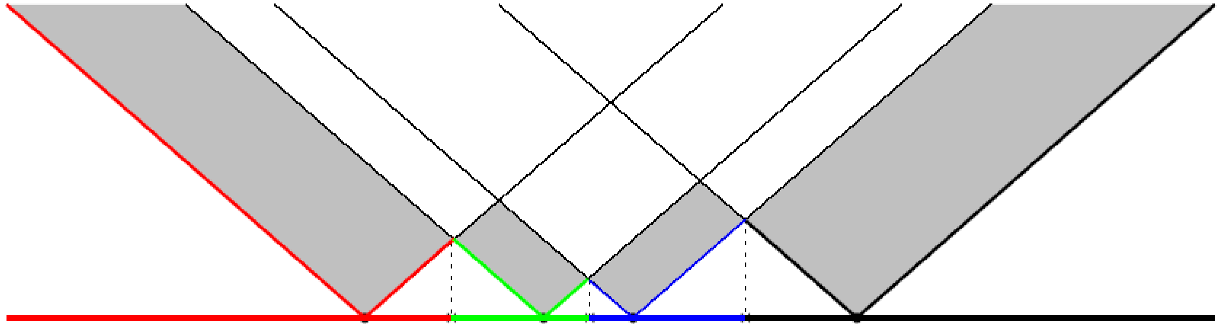
$$\kappa(p) : x_{d+1} = \pi_p(x) = d(x,p)^2 - w(p)$$

So We Can …

- Compute generalized Voronoi diagrams from such cones in one dimension higher
  - Draw cones
  - And see them upward from a very low position
- BUT
  - How can we compute the intersections?

Arrangement!

- We see the lower envelope
- Equivalent to
  the lower boundary of level-1 cells
  of the arrangement of cones



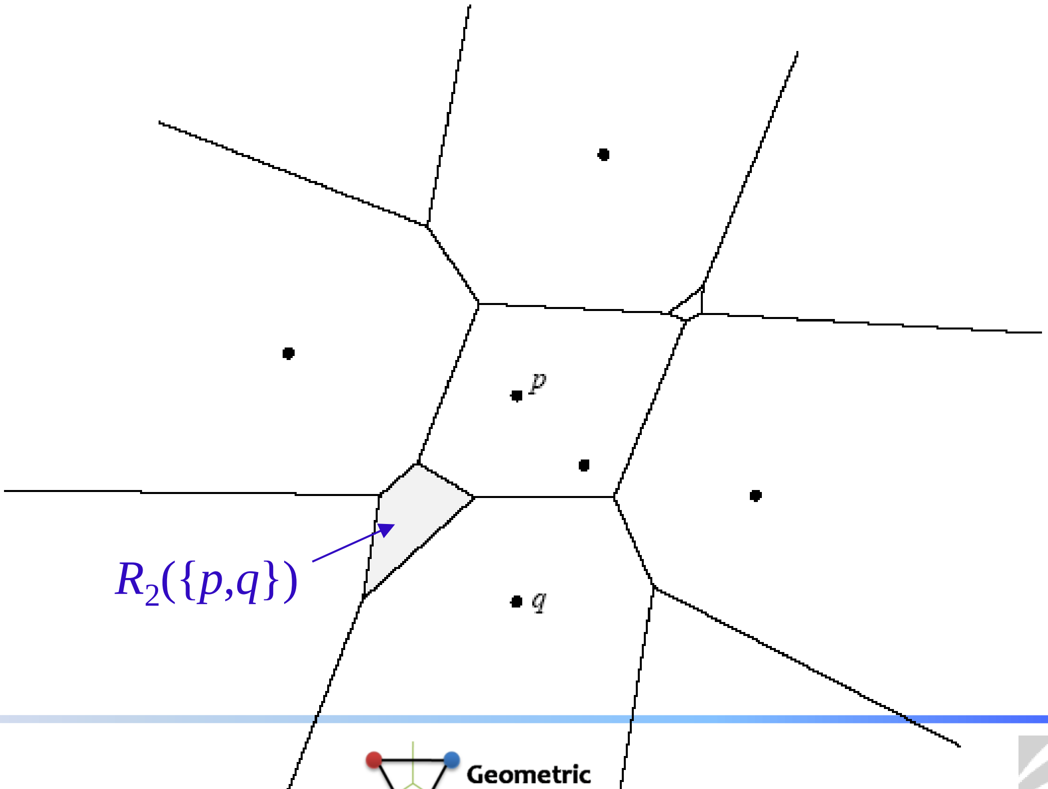
Arrangement

- Voronoi diagrams are projections of level-1 cells onto $x_{d+1} = 0$
- What about level-k cells?

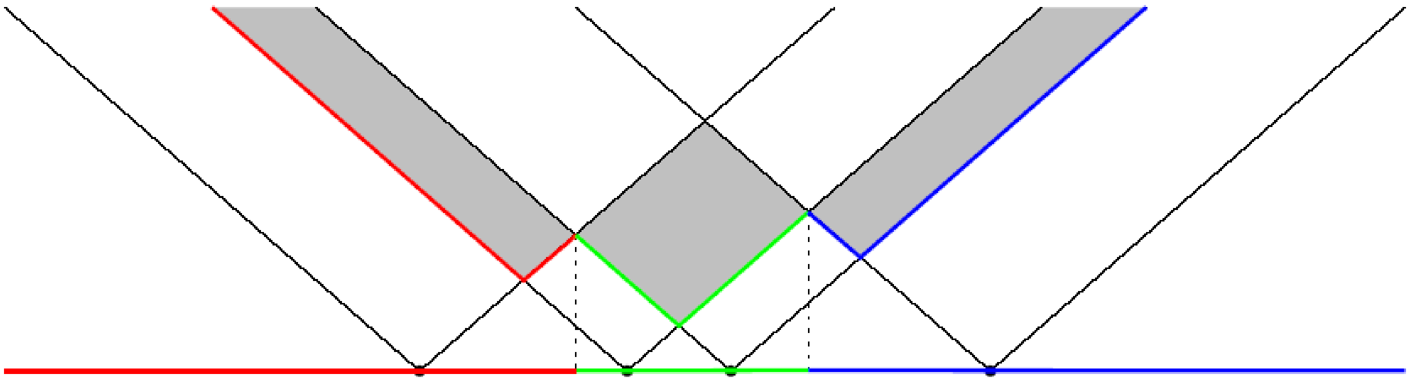
Higher-Order VD

- Order-k Voronoi diagram $V_k(S)$

$$R_k(T) = \{x | p \in T, q \in S \setminus T, d(x,p) < d(x,q)\} \qquad |T| = k$$



Arrangement & Higher-Order VD



- A point (x, x_{d+1}) in a level-k cell,
  - Below it, k cones, say $\kappa(p_1), \ldots, \kappa(p_k)$
  - Above it, n-k cones

$\Leftrightarrow$ x is in $R_k(\{p_1, \ldots, p_k\})$ [EdelsbrunnerSeidel86]

Toward Hyperplanes

- We should compute the arrangement of such cones
- But no efficient algorithms
- Can we simplify such cones or paraboloids?

Toward Hyperplanes

- If $\kappa(p)$ is of the form (a paraboloid)

  $g_p(x) = x_{d+1} = (x-p)^T A(x-p)+t_p$

- Consider the hyperplane

  $f_p(x) = x_{d+1} = <x, a_p> + b_p,$

  where $a_p = -2Ap$, $b_p = p^T Ap + t_p$

- $g_p(x) - g_q(x) = f_p(x) - f_q(x)$
- The collection of functions g and f are order-equivalent.

Toward Hyperplanes

- Levels are relatively defined
- An arrangement of hyperplanes $f_p(x)$ is sufficient enough to compute VDs

For Ordinary VDs

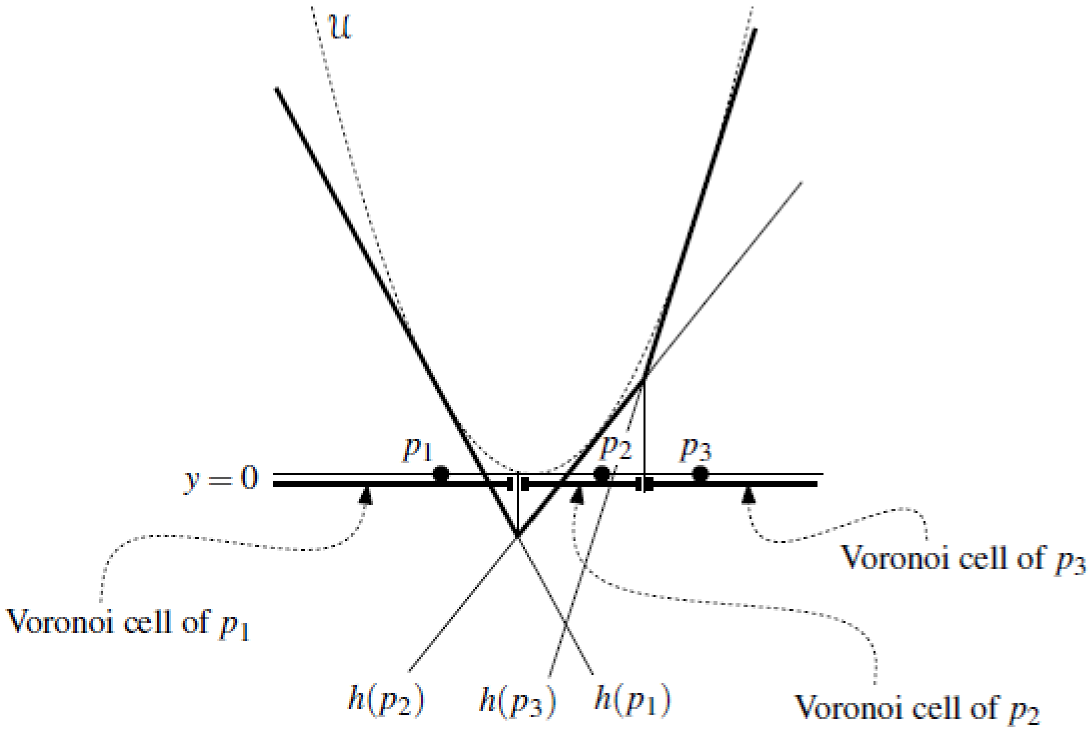
- $\kappa(p) : x_{d+1} = D(x, p)$ is equivalent to

$$\begin{aligned} g_p(x) &= d(p, x)^2 \\ &= (x - p)^T I (x - p) \end{aligned}$$

- $f_p(x) = -2 < p, x > + < p, p >$

Voronoi diagram and half-space intersection

- h(p) : nonvertical plane $z = 2p_x x + 2p_y y - p_x^2 - p_y^2$
- H:= { h(p) | p $\in$ P}
- UE(H) : upper envelope of the planes in H
- Projection of UE(H) on the plane z=0 is the Voronoi diagram of P.



For Power Diagrams

- $$\kappa(p) : x_{d+1} = \pi_p(x) \\ = d(x,p)^2 - w(p)$$

- $$g_p(x) = (x-p)^T I (x-p) - w(p)$$

- $$f_p(x) = -2 < p, x > + < p, p > -w(p)$$

Power diagram

- power diagram, $d = 2$
- $d_p(x)$ can be expressed by the plane in 3D

$$\pi(p): x_3 = 2x^T p - p^T p + w^2(p)$$
$$= x^T x \; - \; d_p(x)$$

- a point x lies in $cell(p)$

$\Leftrightarrow \pi(p)$ is vertically above all other planes $\pi(q)$

- PD(S) is the vertical projection of upper envelope of these planes

Regular Triangulation

- general case ($S \subset \mathbb{R}^d$)

- A site $p \in S$ is transformed into
$$\lambda(p) = \begin{pmatrix} p \\ p^T p - w^2(p) \end{pmatrix}$$
in (d+1)-space

- the projection of the convex hull of $\lambda(S)$ is the regular triangulation of S in $\mathbb{R}^d$

Summary

- An order-k Voronoi (power) diagram can be obtained **from an arrangement** for hyperplanes in one dimension higher

References

- Surveys
  - F. Aurenhammer. Voronoi Diagrams: A Survey of a Fundamental Geometric Data Structure. ACM Computing Surveys, 23(3), 1991.
  - F. Aurenhammer and R. Klein. Voronoi Diagrams. In Handbook of Computational Geometry. Elsevier, 2000.
  - A. Okabe, B. Boots, K. Sugihara and S. N. Chiu. Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, 2nd Ed. John Wiley and Sons, 2000.

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- Key Papers
  - F. Aurenhammer. Power Diagrams: Properties, Algorithms and Applications. SIAM J. Comput., 16(1), 1987
  - H. Edelsbrunner and R. Seidel, Voronoi diagrams and arrangements, Discret. and Comput. Geom., 1, 1986