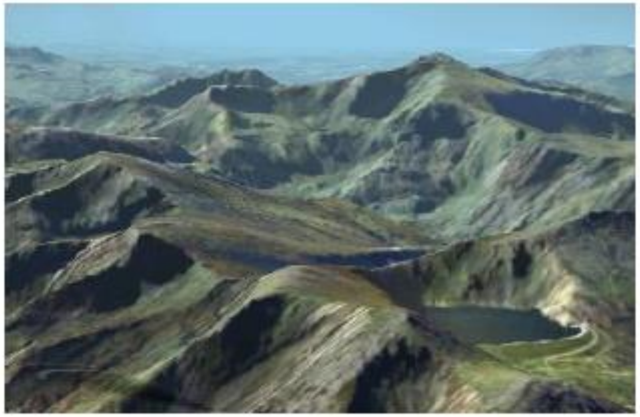
Delaunay Triangulation

Sunghee Choi

Geometric Computing

Terrain

- a terrain is the graph of a function $f: A \subset \mathbb{R}^2 \to \mathbb{R}$
- we know only height value for a set of measurement points
- how can we interpolate the height at other points?
- using a triangulation



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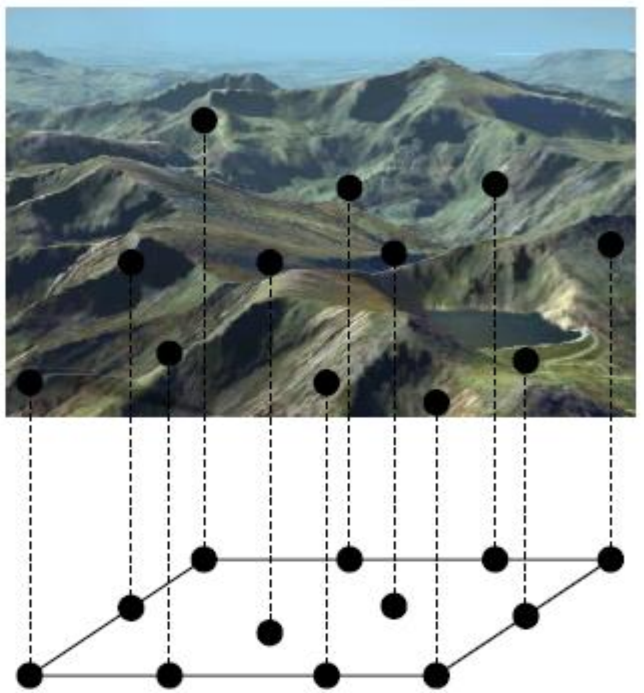

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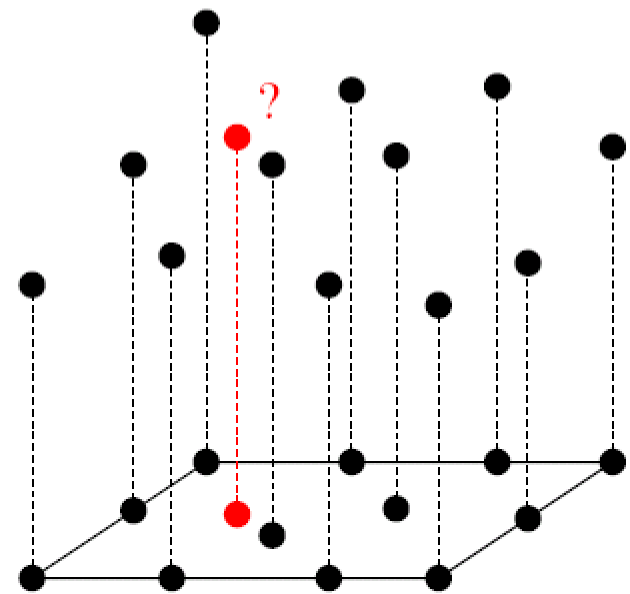

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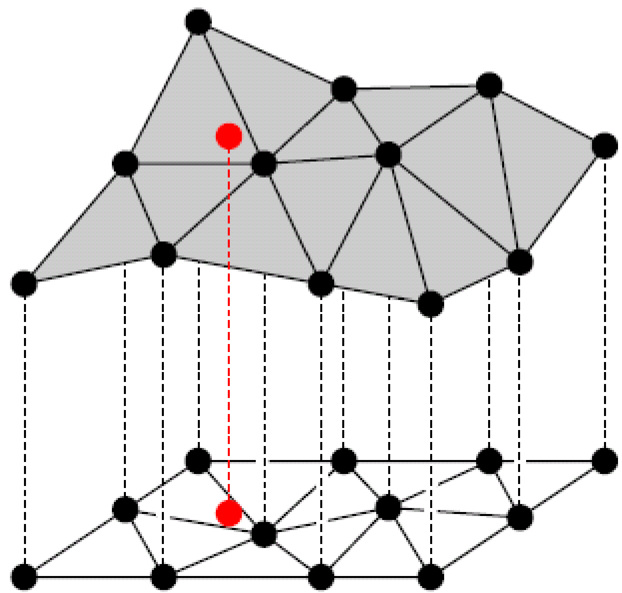
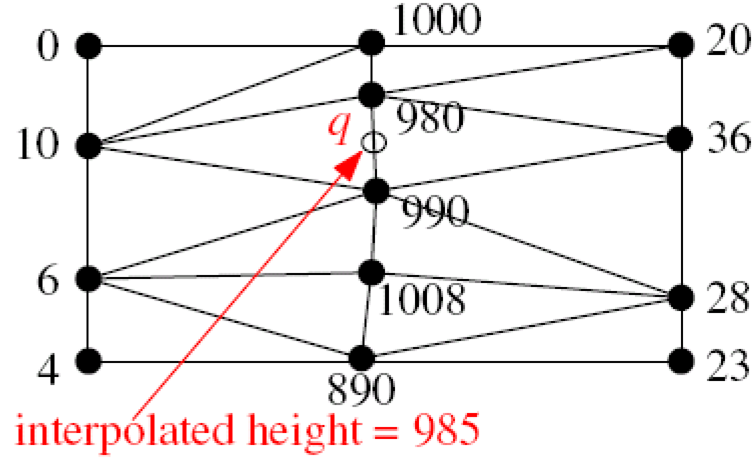
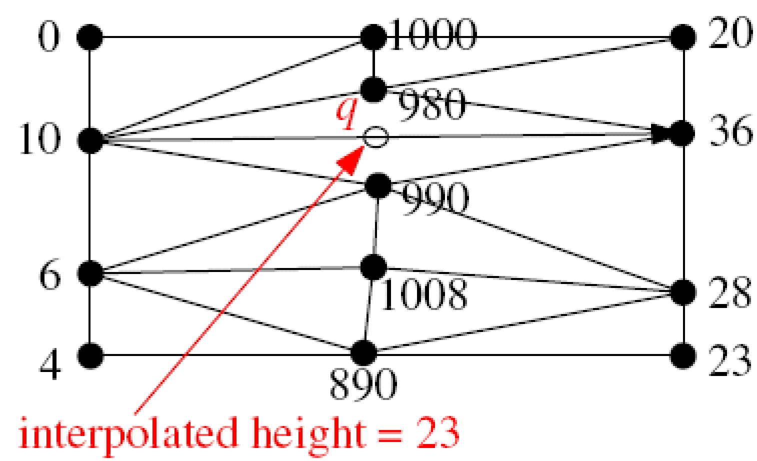


Height Interpolation

- a terrain is the graph of a function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$
- we know only height value for a set of measurement points
- how can we interpolate the height at other points?
- using a triangulation
  - but which one?



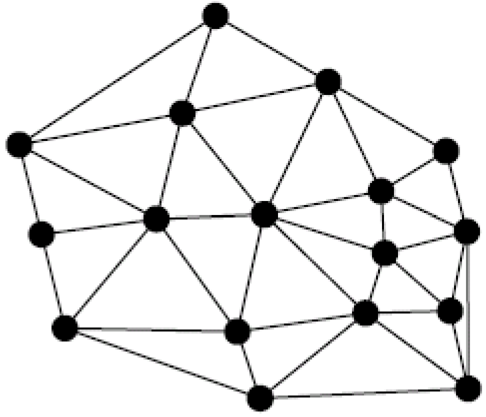


Height Interpolation

- Skinny triangles with small angles are bad!
- We want a triangulation that maximizes the minimum angle.

Triangulation

- Let $P = \{p_1, \dots, p_n\}$ be a point set. A triangulation of P is a maximal planar subdivision with vertex set P.



Triangulation

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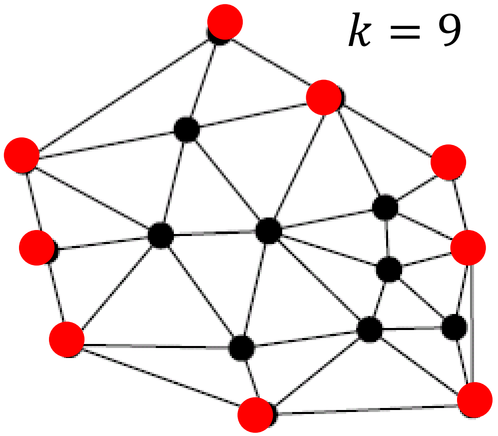
- Complexity

  $2n - 2 - k$ triangles

  $3n - 3 - k$ edges

  where k is the number of points in P on the convex hull of P.



Complexity

- A triangulation with n sites and k sites on the convex hull has 2n-2-k triangles and 3n-3-k edges.

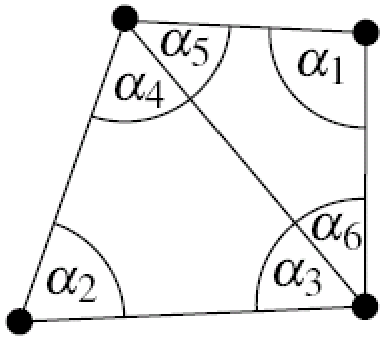
- Pf) Let m denote number of triangles
- number of faces f = m+1 (including the unbounded face)
- Each triangle has 3 edges and the unbounded face has k edges.
- Every edge is incident to 2 faces.
- Total number of edges e = (3m + k)/2
- Euler's formular : n – e + f = 2
- n – (3m+k)/2 + m+1 = 2

Angle Vector

- Let T be a triangulation of P with m triangles and $3m$ vertices. Its *angle vector* is $A(\mathrm{T}) = (\alpha_1, \ldots, \alpha_{3m})$ where $\alpha_1, \ldots, \alpha_{3m}$ are the angles of T sorted by increasing value.

- Let T' be another triangulation of P. We define $\mathrm{A}(\mathrm{T}) > A(\mathrm{T}')$ if $\mathrm{A}(\mathrm{T})$ is lexicographically larger than $\mathrm{A}(\mathrm{T}')$.

- T is *angle optimal* if $\mathrm{A}(\mathrm{T}) \geq A(\mathrm{T}')$ for all triangulation T' of P.



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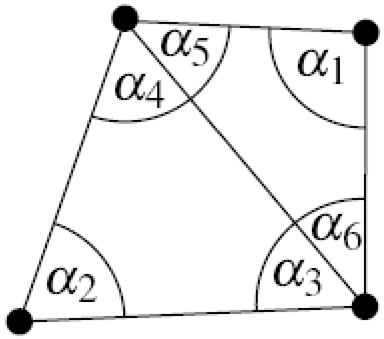

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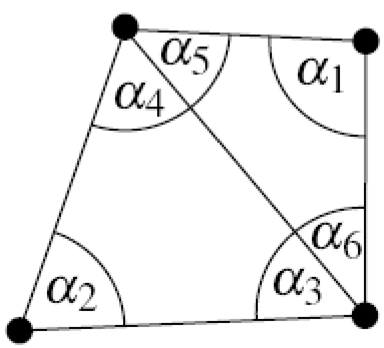

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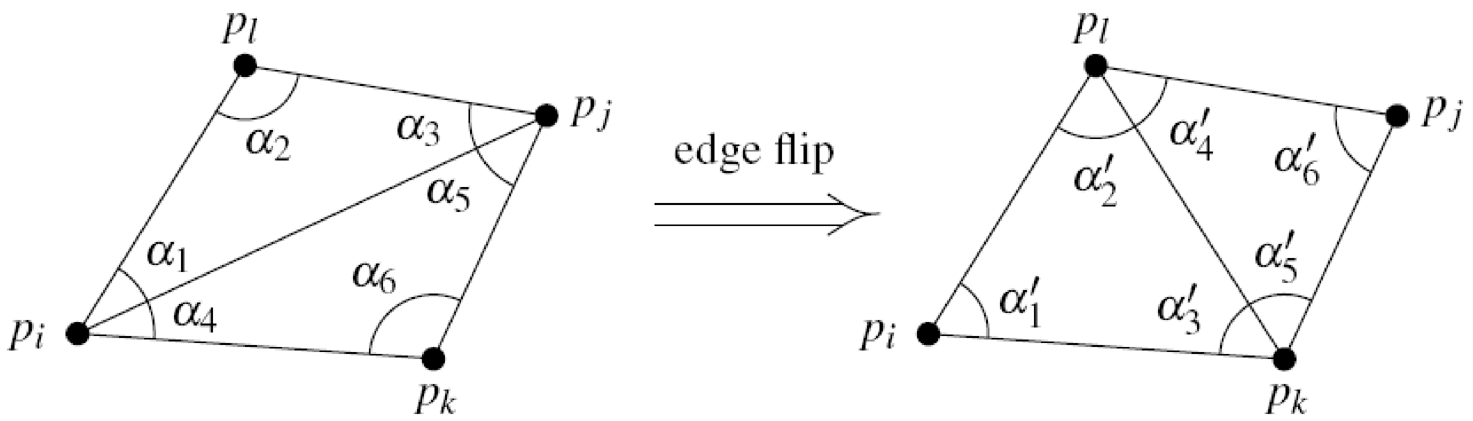
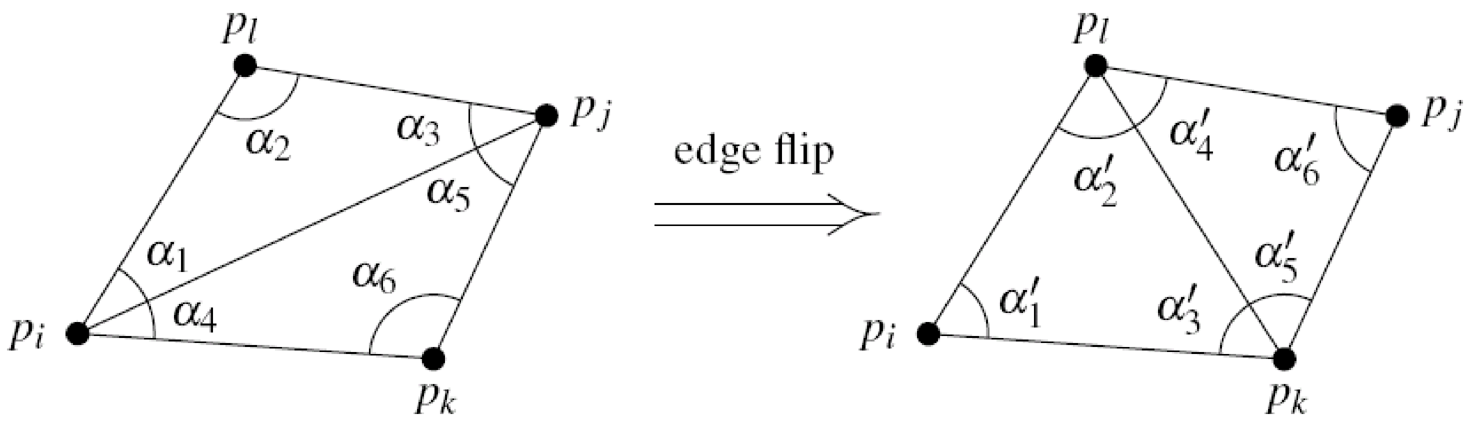
Edge Flipping



- Change in angle vector:
  $\alpha_1, \ldots, \alpha_6$ are replaced by $\alpha'_1, \ldots, \alpha'_6$
- The edge $e = \overline{p_i p_j}$ is *illegal* if $\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i$.
- Flipping an illegal edge-increases the angle vector.

Edge Flipping



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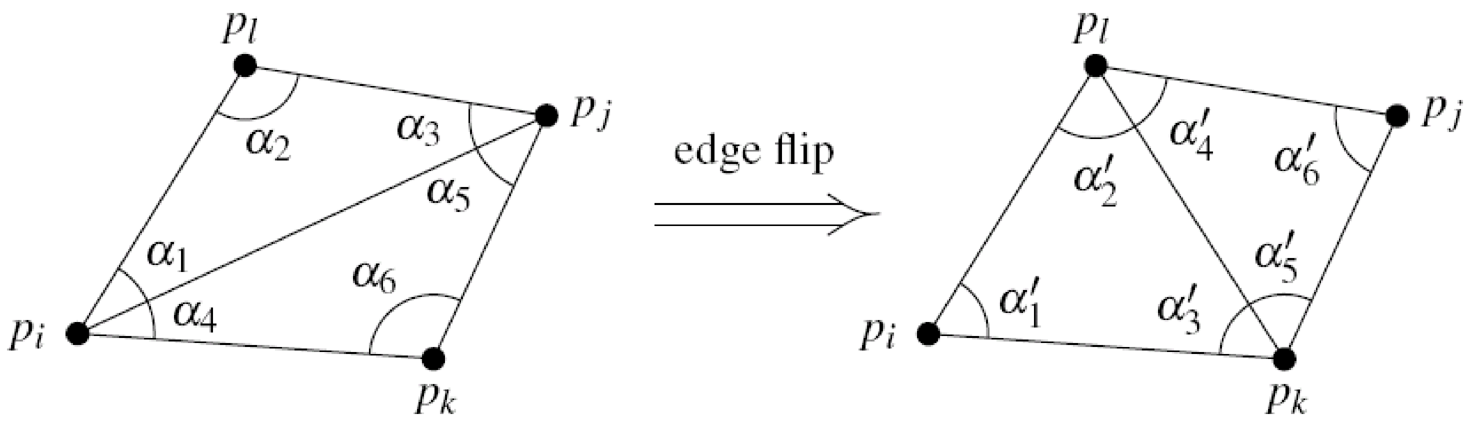

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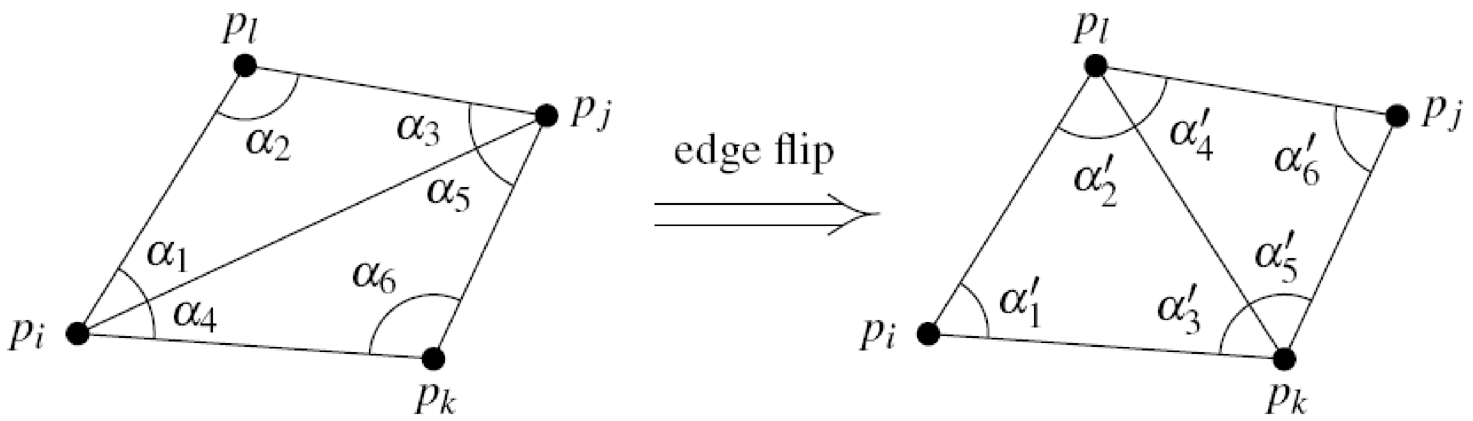

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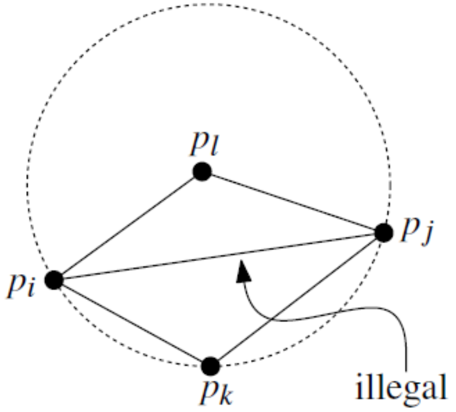
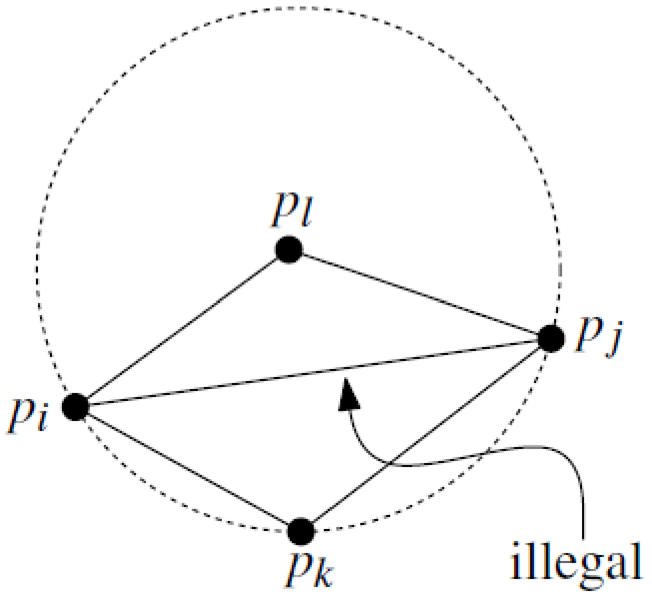
Legal Triangulation

- Lemma : Let edge p_ip_j be incident to triangles $p_ip_jp_k$ and $p_ip_jp_l$ and let C be the circumcircle of $p_ip_jp_k$ The edge is illegal iff p_l lies in the interior of C. Furthermore, if $p_ip_jp_kp_l$ form a convex quadrilateral and do not lie on a common circle, exactly one of p_ip_j and p_kp_l is illegal.
- Pf) by Thales's Theorem



Legal Triangulation

- A legal Triangulation is a triangulation that does not contain any illegal edge.



Legal Triangulation

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Algorithm LEGALTRIANGULATION($\mathcal{T}$)
Input. A triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.
1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_ip_j}$
2. **do** (∗ Flip $\overline{p_ip_j}$ ∗)
3. Let $p_ip_jp_k$ and $p_ip_jp_l$ be the two triangles adjacent to $\overline{p_ip_j}$.
4. Remove $\overline{p_ip_j}$ from $\mathcal{T}$, and add $\overline{p_kp_l}$ instead.
5. **return** $\mathcal{T}$

Legal Triangulation

- Since the angle vector increases in every iteration of the loop and there are final number of different triangulations of P, this algorithm terminates.

Algorithm LEGALTRIANGULATION($\mathcal{T}$)

Input. A triangulation $\mathcal{T}$ of a point set P.

Output. A legal triangulation of P.

1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
2. **do** (* Flip $\overline{p_i p_j}$ *)
3. Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
4. Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
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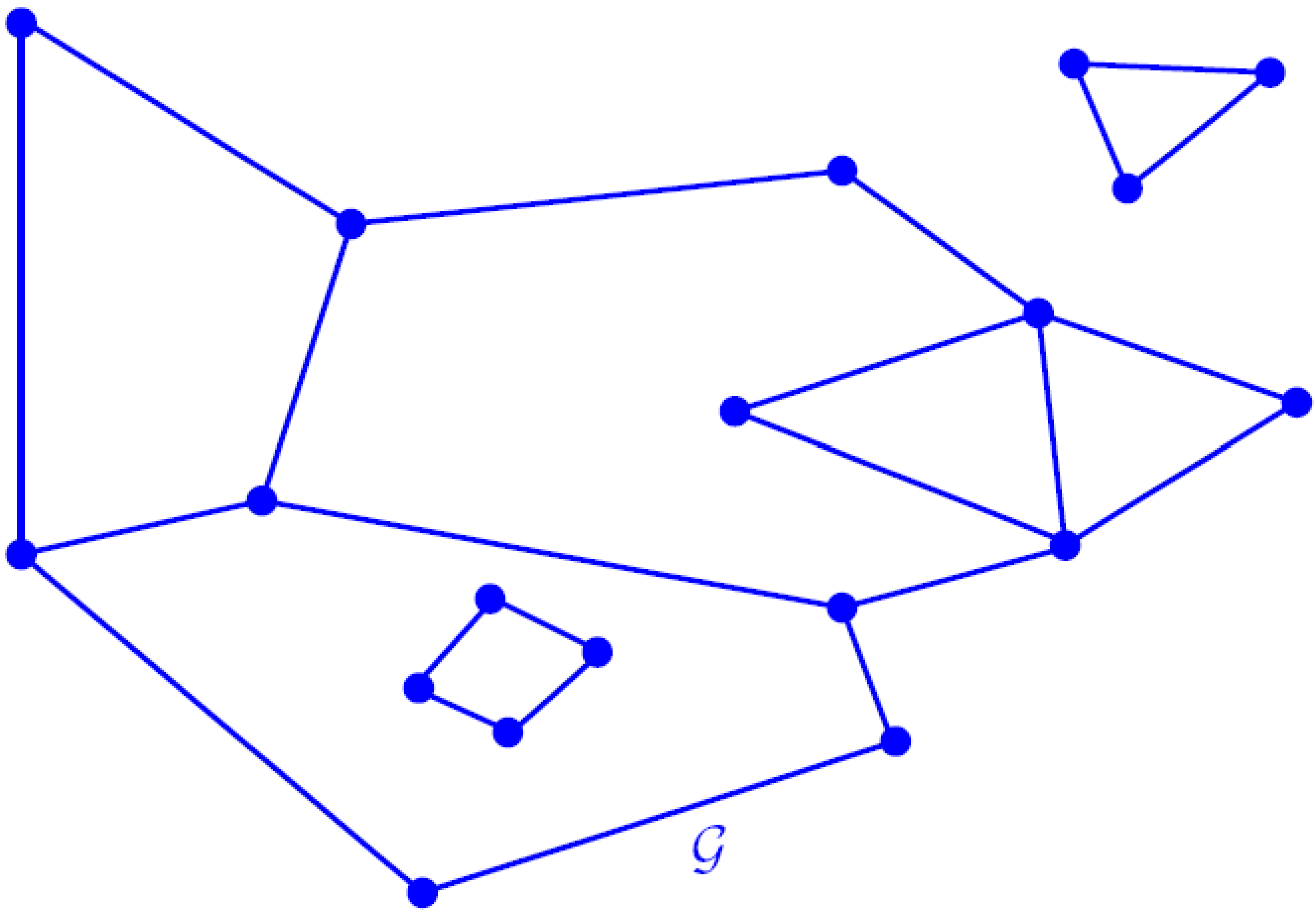
Definition

- What is Delaunay triangulation?

Dual of a planar graph



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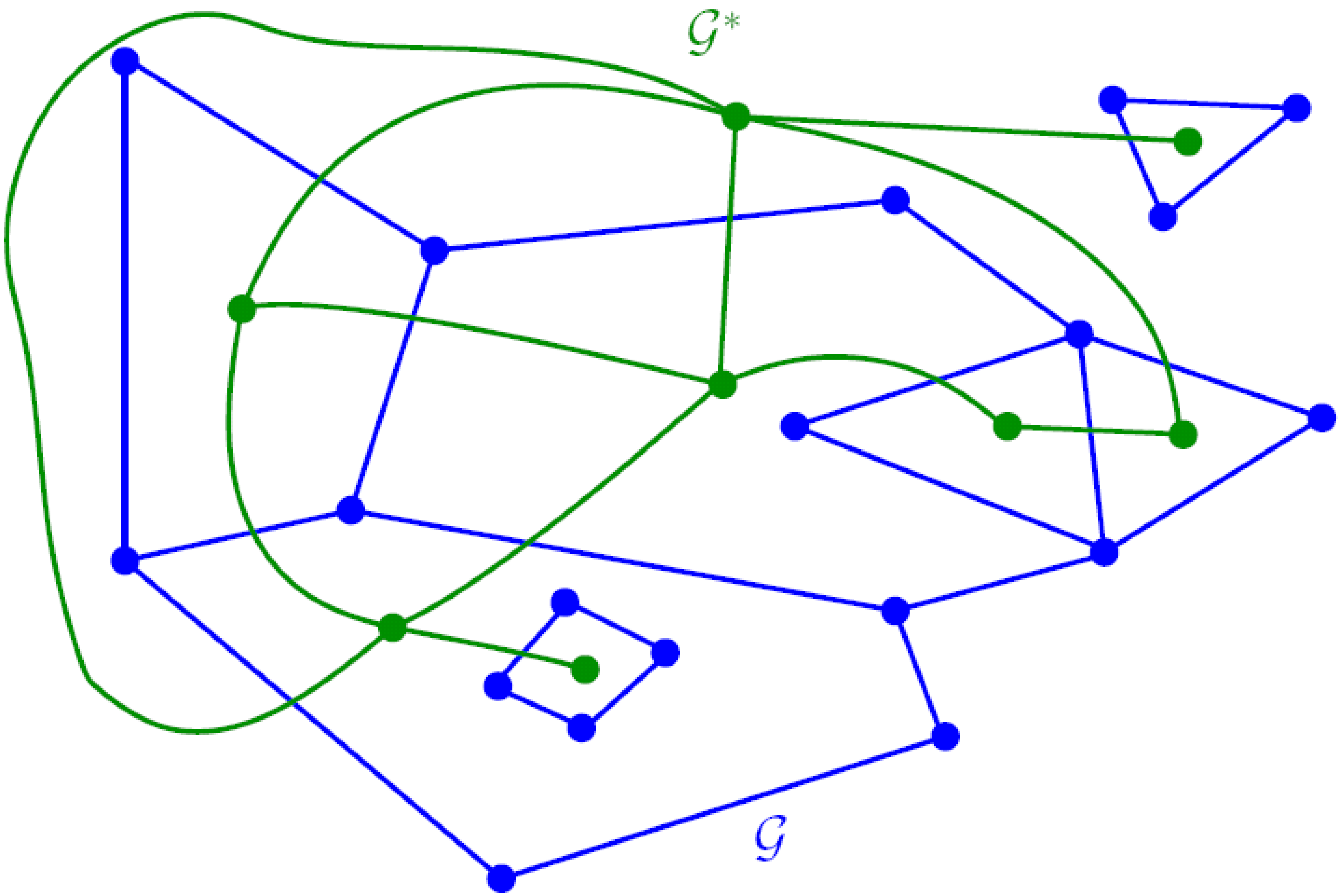
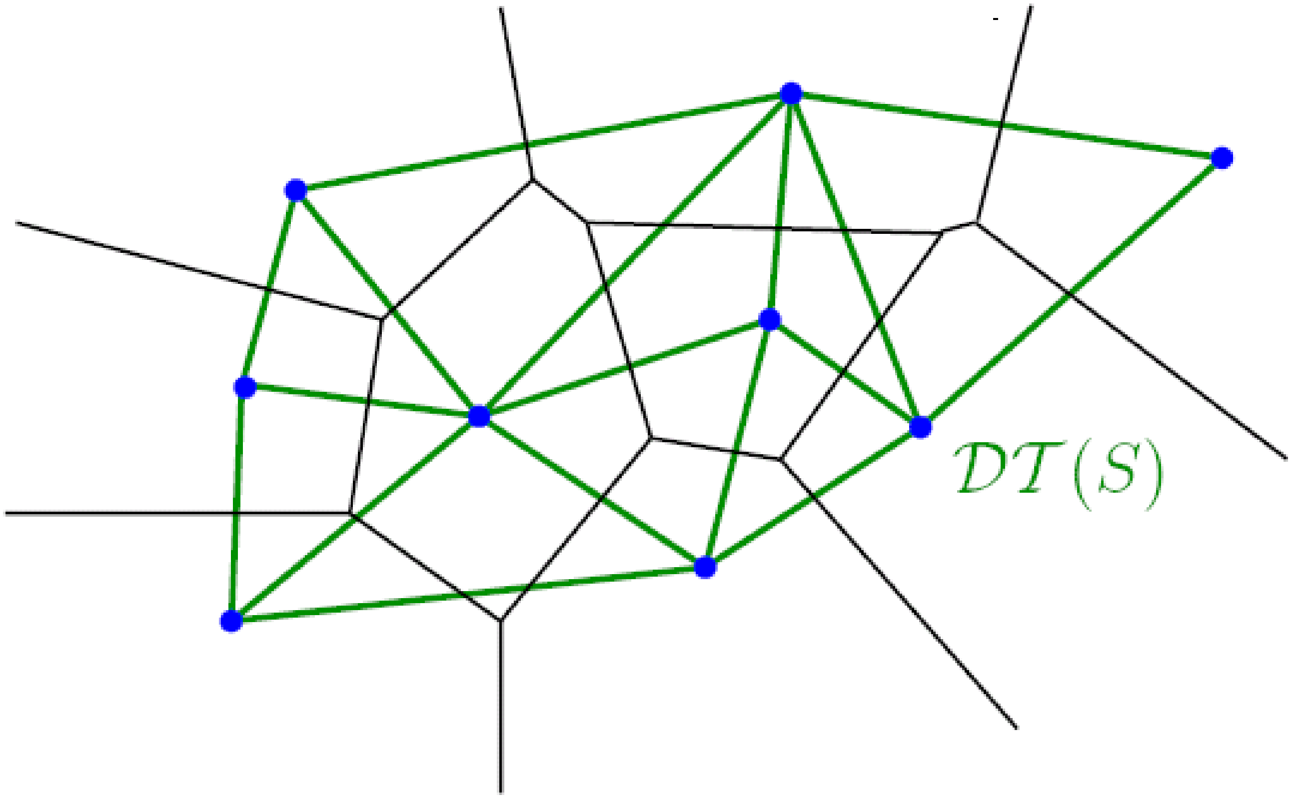
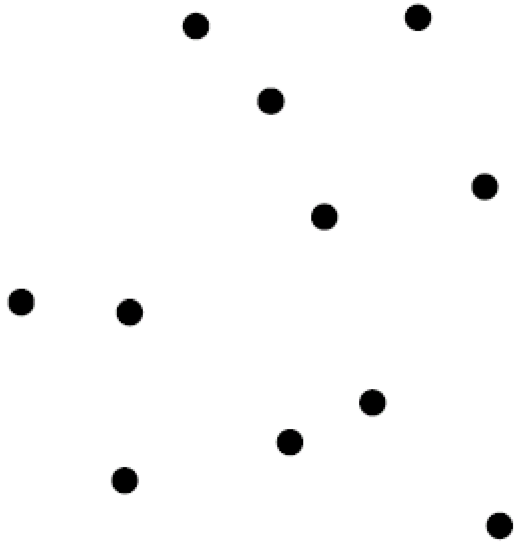


Dual of VD



VD & DT

- Let P be the set of n points in the plane.
- The Voronoi Diagram $Vor(P)$ is the subdivision of the plane into Voronoi cells $V(p)$ for all $p \in P$.
- Let $\mathcal{G}$ be the *dual graph* of $Vor(P)$.
- The Delaunay graph $\mathrm{D}\mathcal{G}(P)$ is the *straight line embedding* of $\mathcal{G}$.



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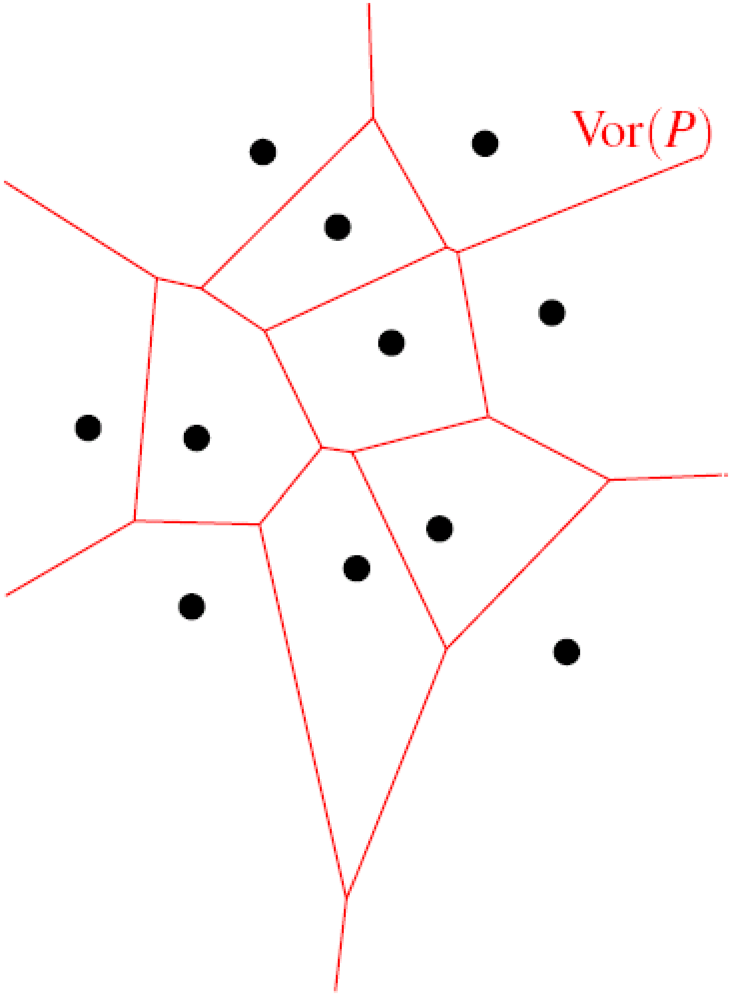

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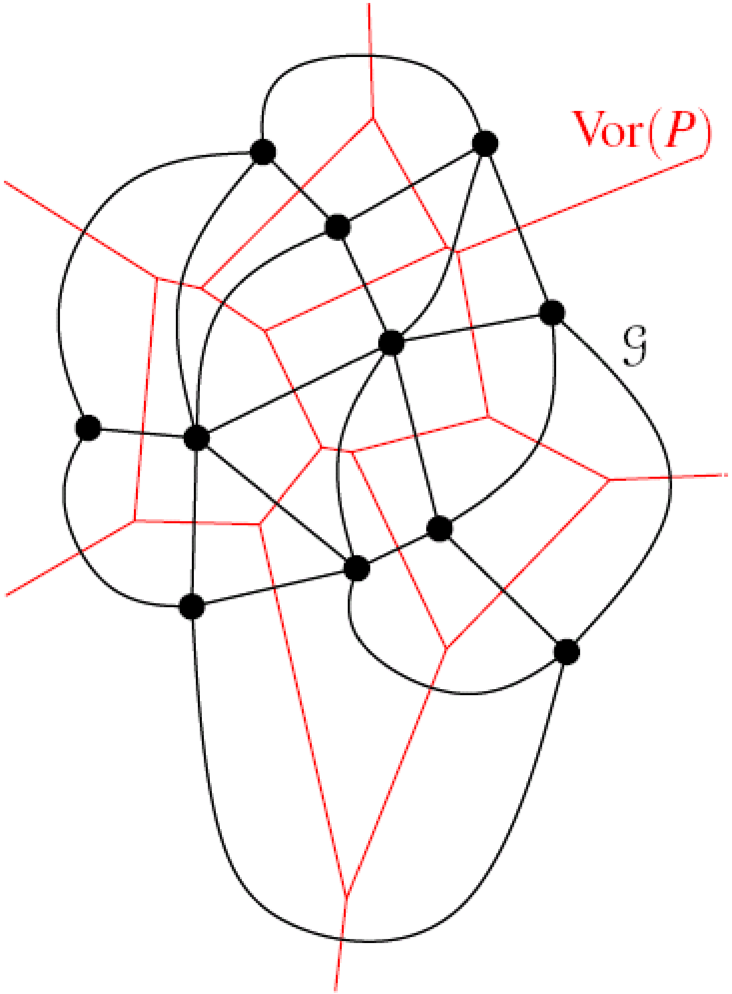
- The Delaunay graph $\mathcal{DG}(P)$ is the *straight line embedding* of $\mathcal{G}$.



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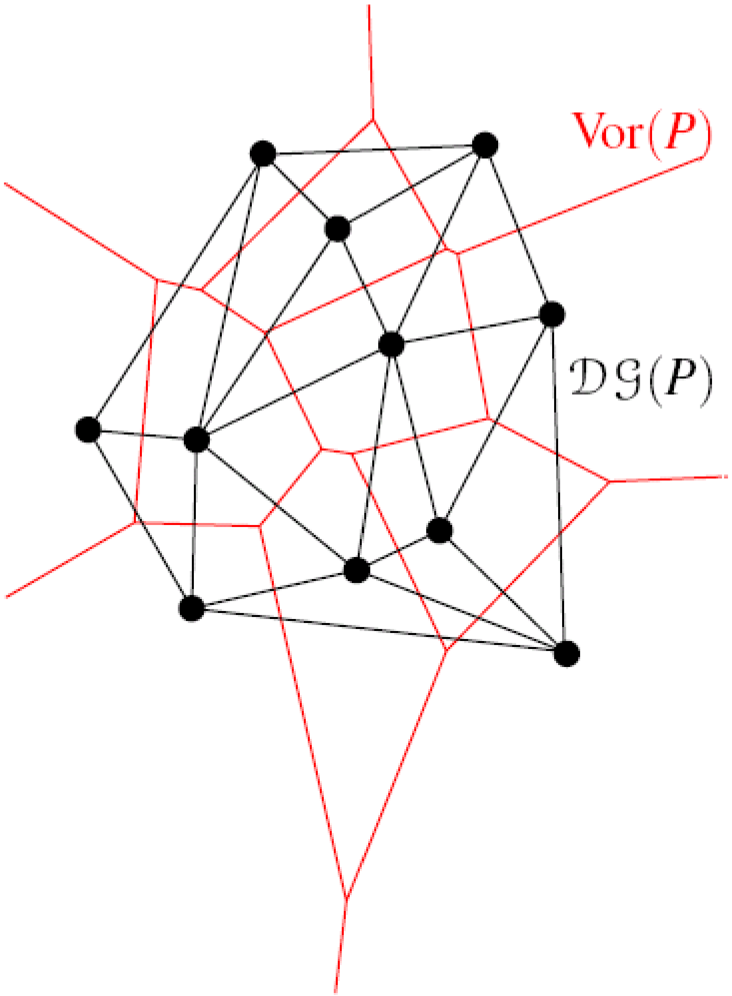
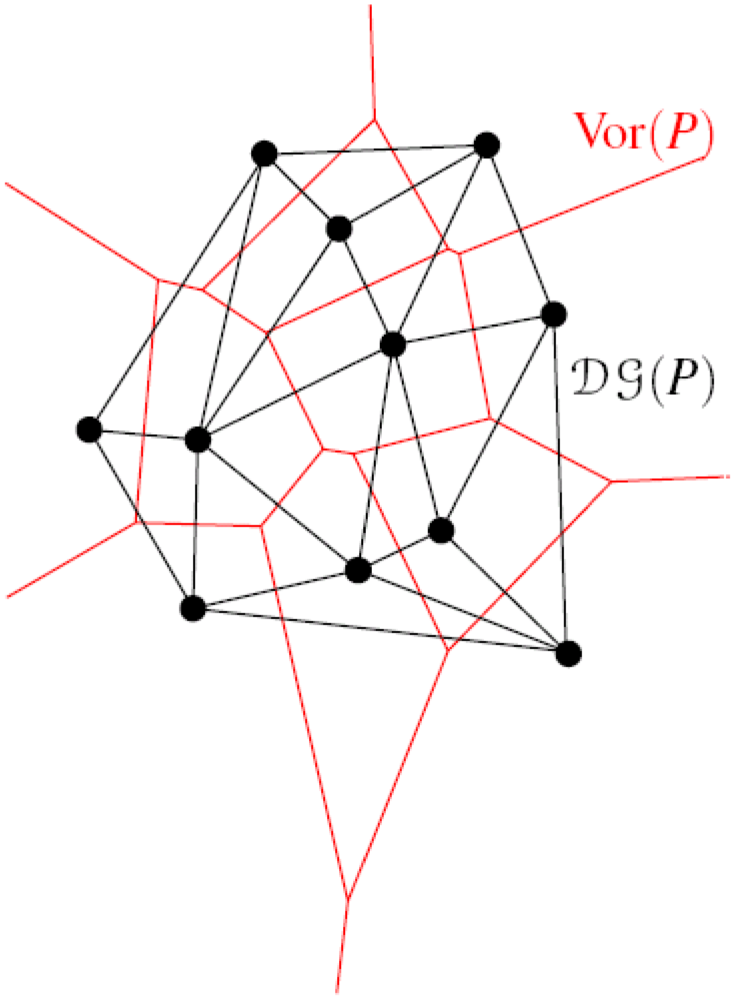


VD & DT

- A set of points is in general position if it contains no four points on a circle and not all points are on a line.
- If P is in general position, then all vertices of the Voronoi diagram have degree three, and all faces of the Delaunay graph are triangles.

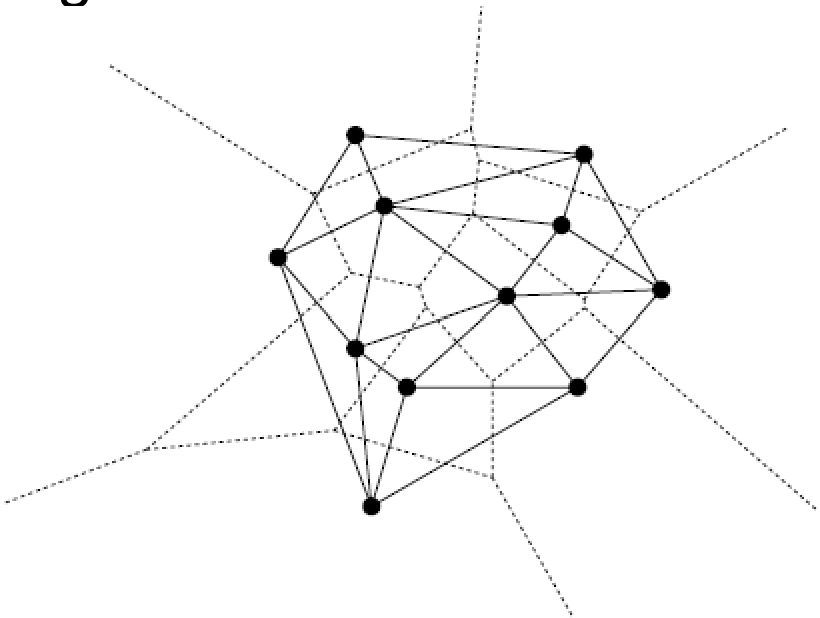


Properties

- Convex hull : the boundary of exterior face of DT is the convex hull.
- Circumcircle property : the circumcircle of any triangle in DT is empty (contains no sites of P).
- Empty circle property : 2 sites are connected by an edge in DT iff there is an empty circle passing through them.
- Closest pair property : the closest pair of sites in P are neighbors in DT.

Circumcircle property

- The circumcircle of any triangle in DT is empty (contains no sites of P).
- Pf) The center of the circumcircle is the corresponding dual Voronoi vertex and 3 sites defining this vertex are its nearest neighbors.



Empty circle property

- 2 sites are connected by an edge in DT iff there is an empty circle passing through them.
- Pf) If 2 sites are neighbors in DT, their Voronoi cells are neighbors. So, for any point on the Voronoi edge between these sites, a circle centered at this point passing through 2 sites cannot contain any other point (since they are closest).
- Conversely, if there is an empty circle passing through them, then the center c of this circle is a point on the edge of the Voronoi diagram between them. Thus, the Voronoi cells of two sites are adjacent, and there is an edge in DT.

Closest pair property

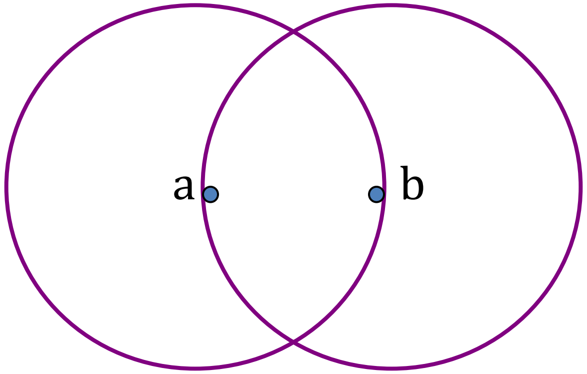
- The closest pair of sites in P are neighbors in DT.
- Pf) Suppose p and q are the closest pair of sites in P.
- The circle having p and q as its diameter is empty.
- Thus, by the empty circle property, they are connected by an edge in DT.

Geometric graphs

- For a given set of sites P, the geometric graph has the sites as vertices, has all edges {a,b} between sites a and b with the distance between a and b as weights.
- Let G(P) be the geometric graph that has edges between every pair of sites.

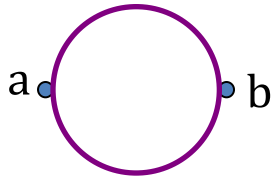
Relative neighbor graph

- RNG(P), the relative neighbor graph, have edges between a and b if no third site is closer to a and to b than d(a,b).



Gabriel graph

- GG(P), the gabriel graph, have edges between a and b when no site is closer than a and b to the midpoint between a and b.

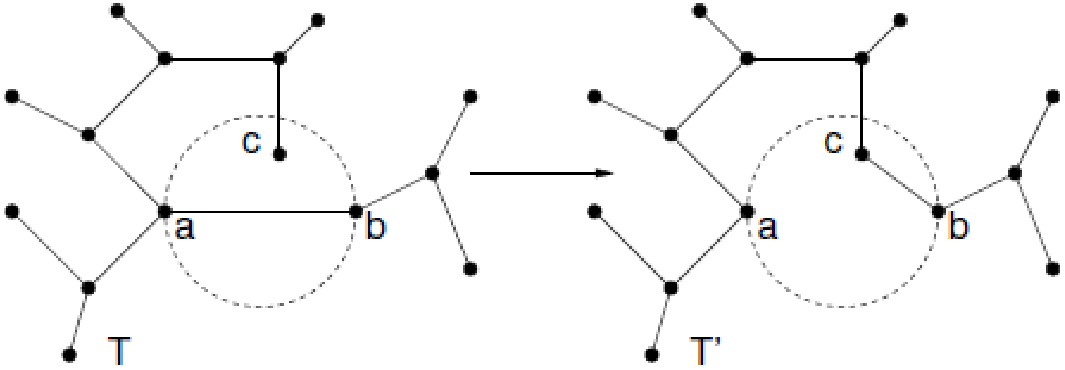


Geometric graphs

- Let EMST(P) be a minimum spanning tree of G(S).
- EMST(P) $\subseteq$ RNG(P) $\subseteq$ GG(P) $\subseteq$ DG(P) $\subseteq$ G(P)

Minimum spanning tree

- Theorem : EMST(P) $\subseteq$ DG(P) (in any dimension!)
- Pf) Let T be the EMST(P) and let w(T) denote its total weight. Let a and b be any two sites that is an edge of T. Suppose to the contrary that ab is not an edge in DT.
- This implies that there is no empty circle passing through a and b, and in particular, the circle whose diameter is the segment ab contains a site, call it c.
- The removal of ab from T splits T into 2 subtrees.
- Assume, w.l.o.g., c lies in the same subtree as a.
- Remove ab and add bc to make a new spanning tree T'.
- w(T') = w(T) + |bc| -|ab| < w(T). Contradiction!



Spanner property

- Given a graph G and parameter t, a *t-spanner* G' of G has the same vertices, and perhaps fewer edges, but the distance between a and b in G' with a factor of t of the distance between a and b in G.
- Theorem [Keil and Gutwin]: DG(S) is a 2.43-spanner of G(S).
- Note that DG(S) has O(n) edges while G(S) has $O(n^2)$ edges.
- Conjecture : DG(S) is a $\pi/2$-spanner of G(S). (not proven!)

Maximizing minimum angle (1/3)

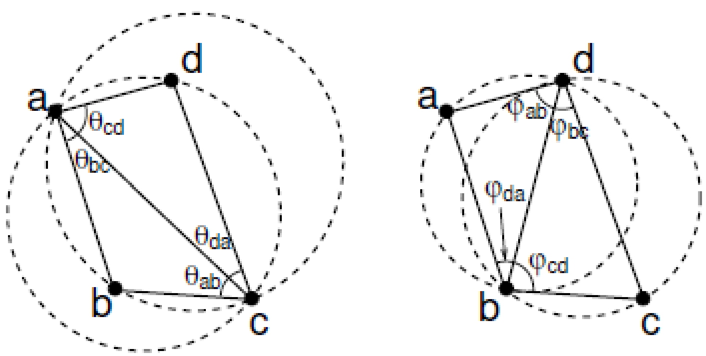
- Among all triangulations, DT maximizes the minimum angle. (Thus, DT tend to avoid skinny triangles.)
- Among all triangulations with the same smallest angle, DT maximizes the second smallest angle, and so on.
- Angle vector : angles sorted in increasing order.
- Theorem : Among all triangulations of a given planar point set, DT has the lexicographically largest angle vector, and in particular, it maximizes the minimum angle.

Maximizing minimum angle (2/3)

- Theorem : Among all triangulations of a given planar point set, DT has the lexicographically largest angle vector, and in particular, it maximizes the minimum angle.
- Pf) Idea : show that for any triangulation that fails to satisfy the empty circle property, it is possible to perform an *edge flip*, which increases the lexicographical sequence of angles.

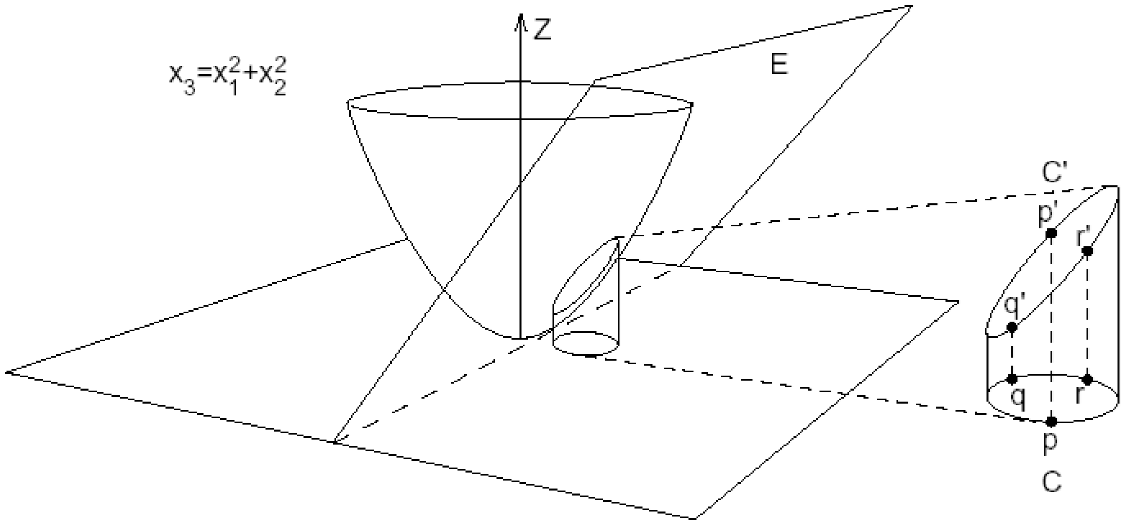
Maximizing minimum angle (3/3)

- Given 2 adjacent triangles abc and cda, s.t. their union forms a convex quadrilateral abcd, the edge flip replaces diagonal ac with bd.
- Suppose that the initial triangle pair violates the empty circle condition (d lies inside the circumcircle of abc). (This implies b lies inside the circumcircle of cda.)
- After flip, the circumcircles of abd and bcd are empty of a,b,c,d, and the minimum angle increases.
- Since there are only a finite number of triangulations, this process terminates with the lexicographically maximum angle vector triangulation and it must satisfy the empty circle condition, hence is the DT.



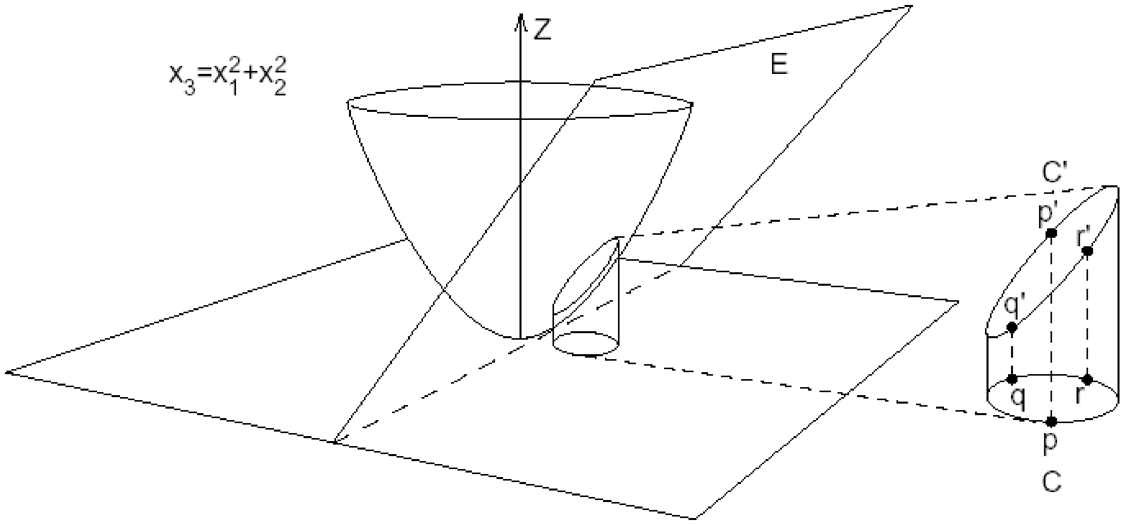
DT and convex hull (1/3)

- Lifting to 3-space
  - $(x_1, x_2) \rightarrow (x_1,\ x_2,\ x_1^2 + x_2^2)$
  - The Delaunay triangulation of S equals the projection onto the (x_1, x_2) -plane of the lower convex hull of S'.



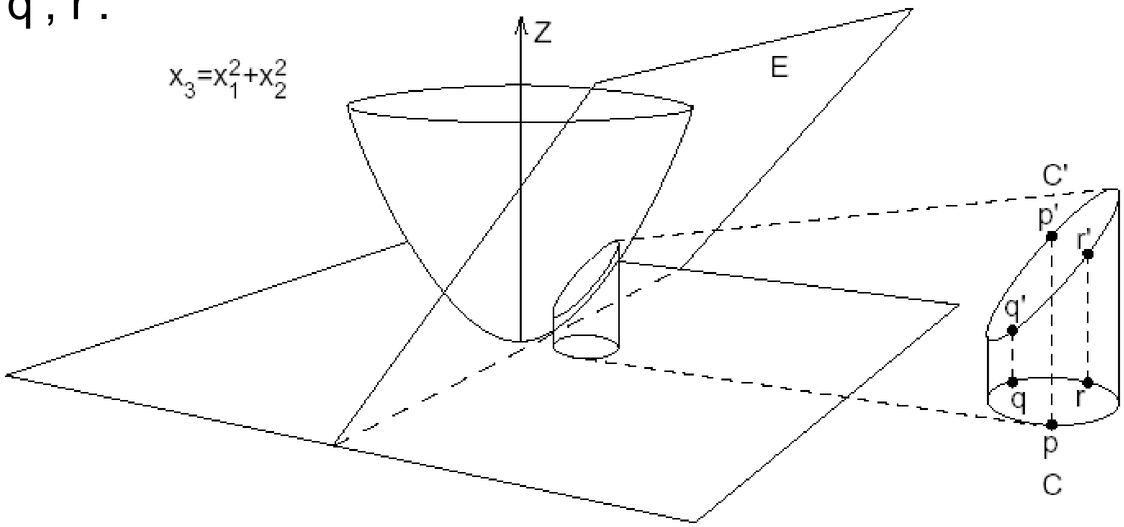
DT and convex hull (2/3)

- Let p,q,r ∈ S, and let p',q',r' denote the lifted points of p,q,r onto the paraboloid.
- Then, p'q'r' define a face of the lower convex hull of S' iff pqr is a triangle of DT of S.



DT and convex hull (3/3)

- Delaunay condition : p,q,r ∈ S form a Delaunay triangle iff its circumcircle is empty.
- Convex hull condition : p',q',r'∈ S' form a face of convex hull of S' iff the plane passing through p', q', r' has all the points of S' lying to one side.
- Lemma : Consider 4 distinct points p,q,r,s in the plane. Let p',q',r',s' be their respective lifted projection on the paraboloid $z = x^2 + y^2$. The point s lies within the circimcircle of p,q,r iff s' lies on lower side of plane passing through p', q', r'.



Applications

- generating good meshes
  http://www.cs.berkeley.edu/~jrs/mesh/
  - skinny triangles are bad in numerical analysis

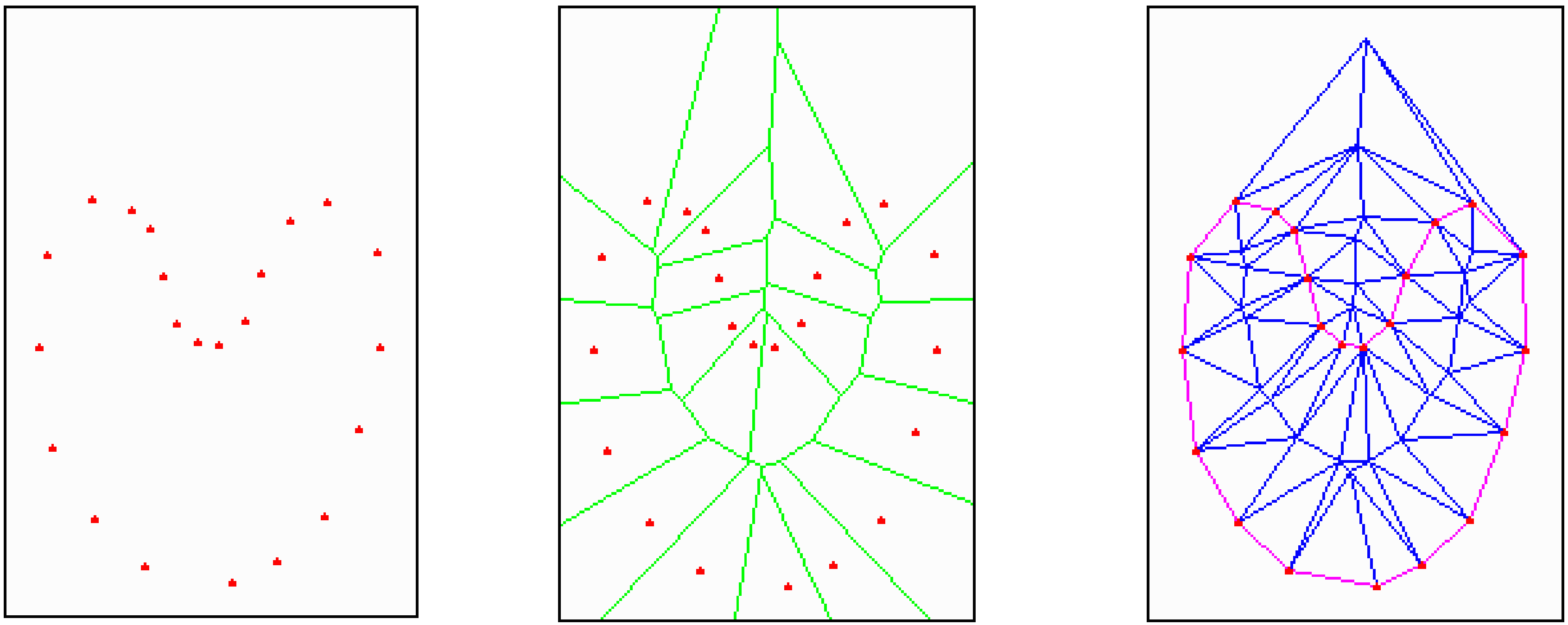
- Interpolation
- Geometric graphs
- Curve/surface reconstruction

Applet

- http://www.cs.unc.edu/~snoeyink/demos/crust/home.html
- http://www.cs.cornell.edu/home/chew/Delaunay.html
- http://valis.cs.uiuc.edu/~sariel/research/CG/applets/Crust/Crust.html

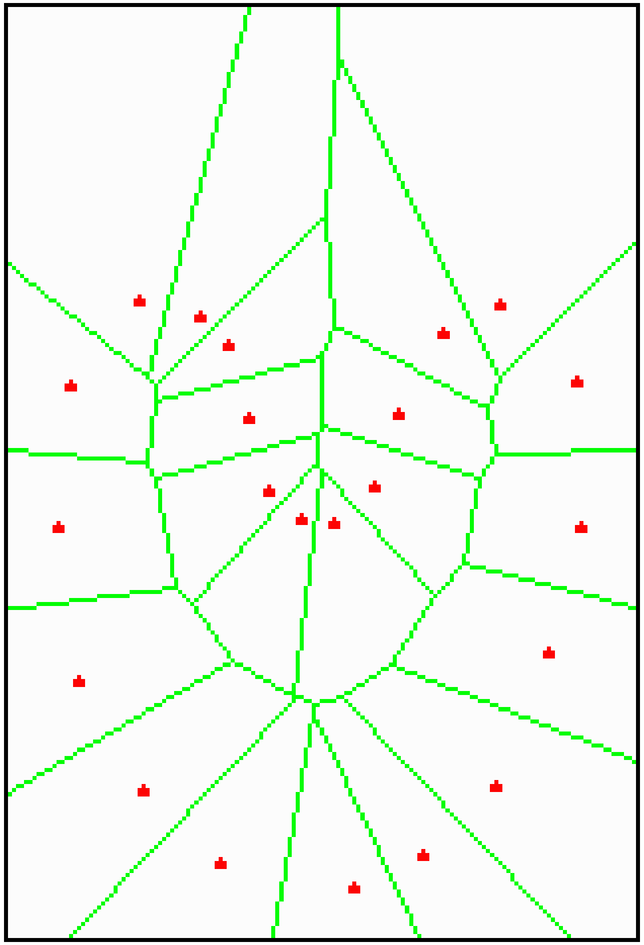
Crust : curve reconstruction

- "The Crust and the Beta-Skeleton : Combinatorial Curve Reconstruction" by Nina Amenta et al.



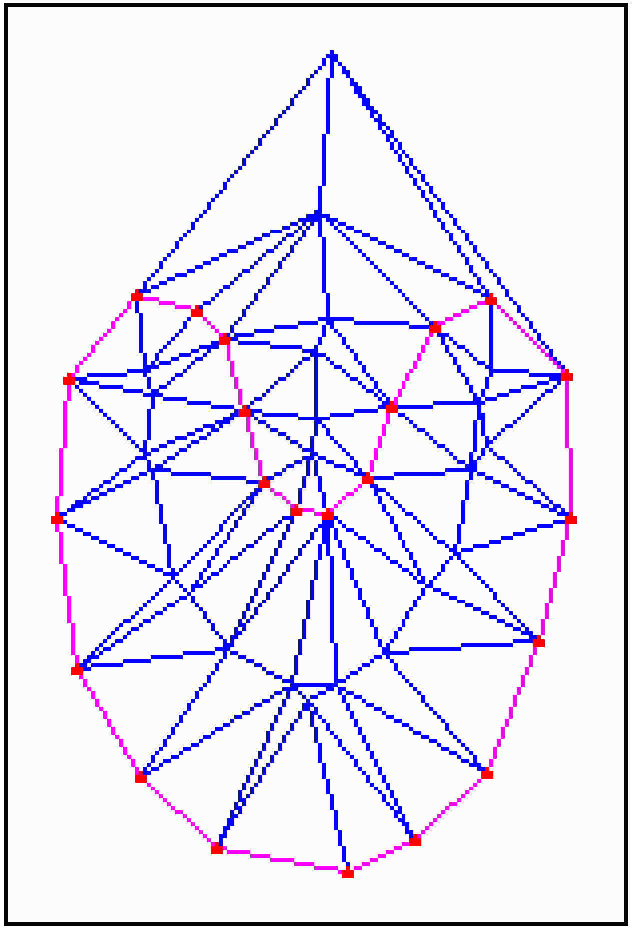
Definition

- Let S be a finite set of points in the plane
- Let V be the vertices of the Voronoi diagram of S.
- An edge between points s1, s2 (both belong to S) belongs to the *crust* of S if there is a disk, empty of points in the union of S and V, touching s1 and s2.



Algorithm

- Let S be a finite set of points in the plane
- Let V be the vertices of the Voronoi diagram of S.
- Let D be the Delaunay triangulation of SUV.
- An edge of D belongs to the crust of S if both its endpoints belong to S.



Algorithms

Voronoi and Delaunay : can compute one from the other in O(n) time. (Delaunay is simpler to compute.)

- edge flipping
- divide-and-conquer
  - O(n log n) worst case
  - split the points in half by a line
  - compute Delaunay triangulation of each piece
  - merge
- sweep
  - O(n log n) worst case
  - use sweep line paradigm in some way
- randomized incremental
  - O(n log n) expected