2013 Spring CS504

Homework #4 Solution

4.14

Worst case : pick elements in decreasing order, which takes $O(n^2)$.

Recurrence equation

$$T(n) = \begin{cases} O(1) & if\ n = 1 \\ \frac{1}{n}(T(n-1) + O(n)) + \frac{n-1}{n}\, T(n-1) & if\ n > 1 \end{cases}$$

So, $T(n) = O(n)$.

4.15

A simple polygon P is star-shaped if and only if the intersection of halfplanes, each of which has its boundary as an edge of P and heading inside of the polygon with respect to that edge is nonempty. We can think the problem as a linear programming problem, with simple objective function (e.g. a height function). So its expected time complexity is linear.

8.1 $p = (a, b)$, $l: y = mx + n$, $p^*: y = ax - b$, $l^* = (m, -n)$

Incident preserving

(=>) $b = ma + n$, so $-n = am - b$.

(<=) $-n = am - b$, so $b = ma + n$.

Order preserving

(=>) b > ma + n , so –n > am – b.

(<=) –n > am – b, so b > ma + n.

8.2 $s \coloneqq \overline{pq}$

The line segment s can be expressed as $\{\, r \mid r = t\,p + (1-t)q,\ 0 \le t \le 1 \}$.

The dual transform of these points are infinite set of lines, $\{r^* | tp^* + (1-t)q^*, 0 \le t \le 1\}$.

It forms a left-right double wedge, which is bounded by $p^* \, and \, q^*$.

8.7

If there exists a separator line l_s for the two sets R and B, then clearly, all the points in R must lie above l_s and all the points in B must lie below l_s or vice versa. Now, consider the duals of the points in R and B, under the duality transform; the corresponding line sets are called R′ and B′ respectively. Let R'_l denote the set of half-planes that are obtained by directing each line in R0 downwards; likewise, let R'_u denote the set of half-planes that are obtained by directing each line in R' upwards. B'_l and B'_u are defined similarly. On account of the order-reversing property of the duality transform, either the point corresponding to l_s is simultaneously below all the lines in R' and above all the lines in B' or vice versa. Thus, either the half-plane set $R'_l \cup B'_u$ has a non-empty intersection or the set $R'_u \cup B'_l$ does.
Not surprisingly, the converse holds as well.

Lemma 1.1 If the half-planes in the set $R'_l \cup B'_u$ have a non-empty intersection, then there exists a separator for the given set of points.
Proof: Pick any point p in the non-empty intersection of $R'_l \cup B'_u$ and focus on its dual p^* in the primal plane. Since $p \in R'_l$, it lies below all the lines in R0 and by the order-reversing property of the duality transform, line p^* must lie above the duals of all the lines in R'. Likewise, since $p \in B'_u$, it lies above all the lines in B' and by the order-reversing property of the duality transform, line p^* must lie below the duals of all the lines in B'. In other words, the line p^* is a separator for the points in R and B.

In identical fashion, we can prove that

Lemma 1.2 If the half-planes in the set $R'_u \cup B'_l$ have a non-empty intersection, then there exists a separator for the given set of points.

From the above discussion, it is clear that the problem of determining the existence of a separator can be reduced to two linear programming problems, which in turn can be solved by the randomized linear-time algorithm discussed in class.